Lecture 2: Counting

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Review

In first lecture, we introduced the "Derivative cocycle"

$$D\colon \mathcal{G}_{\mathcal{F}} \to GL(n, \mathbb{R}) \quad , \quad D_w(h_{\tau_{w,z}}) = Dh_{\tau_{w,z}} \,|_w$$

Maximal exponent along orbits of the geodesic flow gives a decomposition

$$M = M_{\mathcal{H}} \ \cup \ M_{\mathcal{P}} \ \cup \ M_{\mathcal{E}}$$

where the normal derivative for the geodesic flow of $\mathcal{F}$ satisfies:

- $D\varphi_t^{\mathcal{F}}$ is "transversally hyperbolic" on Q for $L_w \subset M_{\mathcal{H}}$
- $D\varphi_t^{\mathcal{F}}$ is bounded (in time) on Q for $L_w \subset M_{\mathcal{E}}$
- $D\varphi_t^{\mathcal{F}}$ has subexponential growth on Q (in time)

How do you tell whether $M_{\mathcal{H}}$ is non-empty? Look at more examples!

Some curious examples

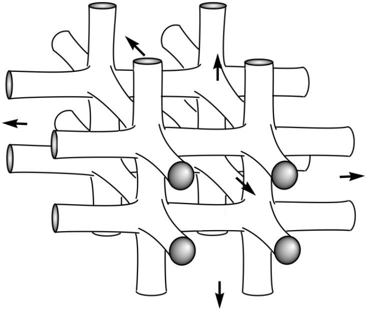
In our first lecture, we started with some "standard" examples.

For this talk, we will start more curious examples.

First, consider the following three examples of complete 2 manifolds, all of which are realized as leaves of foliations of 3-manifolds by "standard constructions".

From the examples, can you guess the dynamics?

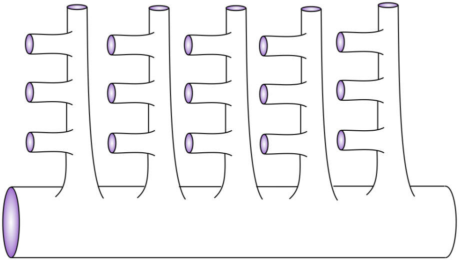
$3 \cdot \omega$



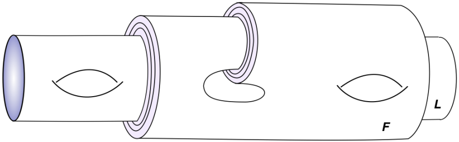
This is called the "Infinite Jungle Gym, appropriately enough.

It is a leaf of a circle bundle over a surface of genus three, where the holonomy consists of three commuting rotations of the circle.

ω^2

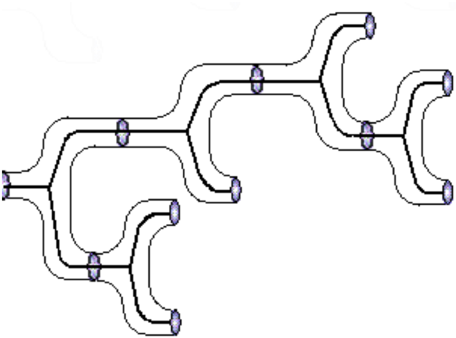


This doesn't have a name, but here is how you get it:



As always, the picture credits go to Lawrence Conlon, circa 1992.

$\aleph_1$

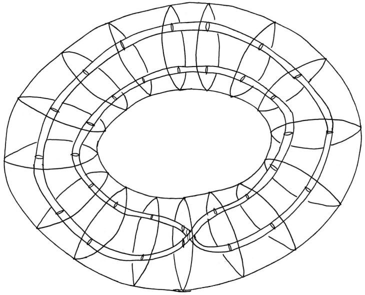


This manifold is said to be “tree-like”.

The Hirsch Construction (circa 1974)

It is the last example that we want to consider more carefully.

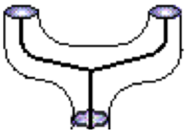
Step 1: Choose an analytic embedding of $\mathbb{S}^1$ in the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ so that its image is twice a generator of the fundamental group of the solid torus. Remove an open tubular neighborhood of the embedded $\mathbb{S}^1$.



The Hirsch Construction, step 2

Step 2: What remains is a three dimensional manifold N_1 whose boundary is two disjoint copies of $\mathbb{T}^2$. $\mathbb{D}^2 \times \mathbb{S}^1$ fibers over $\mathbb{S}^1$ with fibers the 2-disc. This fibration – restricted to N_1 – foliates N_1 with leaves consisting of 2-disks with two open subdisks removed.

Identify the two components of the boundary of N_1 by a diffeomorphism which covers the map $z \mapsto z^2$ of S^1 to obtain the manifold N. Endow N with a Riemannian metric; then the punctured 2-disks foliating N_1 can now be viewed as pairs of pants.



The Hirsch Construction, last step

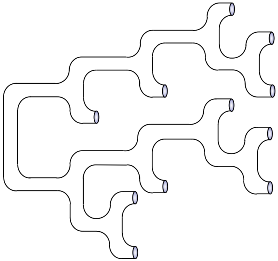
Step 3: The foliation of N_1 is transverse to the boundary, so the punctured 2-disks assemble to yield a foliation of foliation $\mathcal{F}$ on N, where the leaves without holonomy (corresponding to irrational points for the chosen doubling map of S^1) are infinitely branching surfaces, decomposable into pairs-of-pants which correspond to the punctured disks in N_1.

The curious point is that this works for any covering map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the doubling map along a meridian.

In particular, as Hirsch remarked in his paper, the proper choice of such a map results in a codimension-one, real analytic foliation, such that all leaves accumulate on an exceptional minimal set.

Instability in the geodesic flow

The Hirsch foliation always has a leaf as follows:



Consider the behavior of the geodesic flow, starting at a "bottom point" $w \in L_w$. For a each radius $R \gg 0$, the terminating points of the geodesic rays of length at most R will "jump" between μ^R ends, for some $\mu > 1$.

We want to count this μ, generically!

Orbits

Recall the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ constructed in Lecture 1, modeled on a complete transversal $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k$ associated to a finite covering of M by foliations charts. Given $w \in \mathcal{T}$ and $z \in L_w \cap \mathcal{T}$ and a leafwise path $\tau_{w,z}$ joining them, we obtain an element $h_{\tau_{w,z}} \in \mathcal{G}_{\mathcal{F}}$.

Definition: The orbit of $w \in \mathcal{T}$ under $\mathcal{G}_{\mathcal{F}}$ is

$$\mathcal{O}(w) = L_w \cap \mathcal{T} = \{z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_{\mathcal{F}}, w \in Dom(g)\}$$

The second description allows us to decompose the orbit into "periods".

Lengths of orbits

For $g \in \mathcal{G}_{\mathcal{F}}$ we say that $\|g\| \leq d$, if g can be expressed as a product of at most d maps obtained as the holonomy of adjacent open charts:

$$g = h_{i_0,i_1} \circ h_{i_1,i_2} \circ \cdots \circ h_{i_{d-1},i_d} | Dom(g)$$

where $U_{i_{\ell-1}} \cap U_{i_\ell} \neq \emptyset$ for all $1 \leq \ell \leq k$.

The groupoid norm $\|\gamma_w\| = d$, if d is the least such integer such that there exists $g \in \mathcal{G}_{\mathcal{F}}$ with germ $\gamma_w = [g]_w$ and $[g]_w \leq d$. The norm of the identity is always 0.

Define the “orbit of radius d in the groupoid word norm” to be:

$$\mathcal{O}_d(w) = \{z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_{\mathcal{F}}, w \in Dom(g), \|[g]_w\| \leq d\}$$

Growth types of orbits

Definition: The growth function of an orbit is $Gr(w,d) = \#\mathcal{O}_d(w)$.

Of course, this depends upon almost all choices! However, its "growth type function" is an independent of all choices, as observed by Plante in 1975.

Definition: $w \in \mathcal{T}$ has *exponential orbit growth type* if $Gr(w,d)$ behaves like an exponential function of d; *polynomial growth type* if it behaves like a polynomial function of d; and *subexponential* if dominated by every exponential function of d.

Growth of groups

Growth functions for finitely generated groups are a basic object of study in geometric group theory in recent years.

Let $\Gamma = \langle \gamma_0 = 1, \gamma_1, \ldots, \gamma_k \rangle$. Then $\gamma \in \Gamma$ has norm

$$\|\gamma\| \leq d \iff \gamma = \gamma_{i_1}^{\pm} \cdots \gamma_{i_d}^{\pm}$$

$$\Gamma_d \equiv \{\gamma \in \Gamma \mid \|\gamma\| \leq d\}$$

The growth function $Gr(\Gamma, d) = \#\Gamma_d$ depends upon the choice of basis for Γ, but its growth type does not.

Perhaps most famous theorem of Gromov:

Theorem: Suppose Γ has polynomial growth type for some generating set. Then there exists subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' is a nilpotent group.

Growth of leaves

The homogeneity of groups makes their growth functions "amenable" to study - the growth rate is the same for balls in the word metric about any point $\gamma_0 \in \Gamma$.

For foliation pseudogroups, this is one of the basic open questions:

Problem: How does the class of the function $w \mapsto Gr(w, d)$ behave, as a Borel function of $w \in \mathcal{T}$?

Examples of Ana Rechtman show that even for smooth foliations of compact manifolds, this function is not uniform as function of $w \in \mathcal{T}$.

Very surprising!

Uniform growth of leaves

One of the first "classifying" results about the measurable orbit equivalence type of foliations:

Theorem:[Series 1977] If the growth type of all functions $Gr(w, d)$ are polynomial, then the equivalence relation on $\mathcal{T}$ defined by $\mathcal{G}_{\mathcal{F}}$ is hyperfinite.

A hyperfinite foliation is the measurable limit of Borel equivalence relations with finite orbits.

Theorem: [Dye 1957] A foliation defined by a non-singular flow is always hyperfinite.

Theorem: [Connes-Feldman-Weiss 1982] If $\mathcal{F}$ is defined by the suspension of the action of a finitely generated group Γ, where Γ is amenable, then the equivalence relation on $\mathcal{T}$ it defines is hyperfinite.

Topological classification

Problem: Is there a special subclass of minimal hyperfinite foliations which can be topologically classified?

Maybe...

In the late 1970's and early 1980's, Cantwell & Conlon, Hector, Nishimori, Tsuchiya in particular studied the case of codimension one foliations.

The result was the "theory of levels" for foliations of class C^2, a form of generalized Poincaré-Bendixson Theory for leaves of foliations.

For real analytic foliations, the results are very satisfying, regarding theory for foliations with all leaves of polynomial growth. Closest approximation to a generalized form of Gromov's Theorem above.

But in general? There is no theory for C^1-foliations of codimension-one, for example. Gilbert Hector was too busy making up *nasty* examples...

Topological dynamics

One approach to classification if to impose restrictions on the dynamics.

Definition: A pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ is *proximal* if there exists $\delta > 0$ such that for all $w, w' \in \mathcal{T}$ with $d_{\mathcal{T}}(w, w') < \delta$, then for all $\epsilon > 0$ there exists $h_{\tau w,z} \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(h_{\tau w,z})$ and $d_{\mathcal{T}}(h_{\tau w,z}(w), h_{\tau w,z}(w')) < \epsilon$.

Definition: A pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ is *distal* if it is not proximal.

Definition: A pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ is *equicontinuous* if there exists a metric $d'_{\mathcal{T}}$ on $\mathcal{T}$ equivalent to the Riemannian distance function, such that for all $w, w' \in \mathcal{T}$ and $h_{\tau w,z} \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(h_{\tau w,z})$,

$$d'_{\mathcal{T}}(h_{\tau w,z}(w), h_{\tau w,z}(w')) = d'_{\mathcal{T}}(w, w')$$

These are very old concepts, dating from 1930's, and extensively studied for topological group actions in 1950's and 1960's.

Equicontinuous matchbox manifolds

Definition: A minimal set $\mathfrak{M} \subset M$ for a foliation $\mathcal{F}$ is exceptional if the intersection $\mathfrak{M} \cap \mathcal{T}$ is a Cantor set.

Thus, $\mathfrak{M}$ is a foliated space, as studied by A. Candel and L. Conlon in Chapter 11, of their text **Foliations, I**.

Exceptional minimal sets are special, though, as such $\mathfrak{M}$ are *transversally* zero-dimensional. Such foliated spaces are called "matchbox manifolds" in the topological dynamics literature.

Theorem: [Clark-Hurder 2009] If $\mathfrak{M} \subset M$ is an *exceptional minimal set* for a foliation (that is, $\mathfrak{M}$ is transversally a Cantor set), and the dynamics of $\mathcal{F}$ restricted to $\mathfrak{M}$ are equicontinuous, then $\mathfrak{M}$ is homeomorphic as a foliated space to a generalized solenoid. $\Longrightarrow$ $\mathcal{F}$ on $\mathfrak{M}$ is hyperfinite.

Problem: Can the distal matchbox manifolds be classified?

Growth and dynamics

One problem with the previous "counting argument", especially as concerns the Hirsch example, is that it just counts the number of times the leaf crosses a transversal $\mathcal{T}$ in fixed distance, but does not take into account whether these crossings are "nearby" or "far apart".

There are Riemannian foliations with all leaves of exponential growth type.

In the Hirsch examples, the handles at the end of each "ball of radius d" appear to be widely separated transversally, so somehow this is different.

Hirsch examples, revisted

The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After d-iterations, the inverse map to $z \mapsto z^{2^d}$ has derivative of norm 2^d.

Thus, for a Hirsch foliation modeled on this map, every leaf is transversally hyperbolic.

How to take this into account?

Introduce foliation "geometric entropy" of Ghys, Langevin and Walczak!

Next time...

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