Lecture 3: Exponential Complexity

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Recall a simple example from advanced calculus.

Let $f(x) = x/2$.

Let $g(x)$ be smooth with $g(0) = 0$, $g'(0) = 1/2$.

Then $g \sim f$ near $x = 0$. That is, for $\delta > 0$ sufficiently small, there is a smooth map $h: (-\epsilon, \epsilon) \to \mathbb{R}$ such that $h^{-1} \circ g \circ h = f(x)$ for all $|x| < \delta$.

**Fact:** Exponentially contracting (or simply hyperbolic) maps have a single parameter (their derivative $g'(0)$ at the fixed point) for their germinal conjugacy class.

For maps which are “completely flat” at the origin, where $g(0) = 0$, $g'(0) = 1$, $g^k(0) = 0$ for all $k > 1$, no such classification exists.

**Moral:** *Complexity is Simplicity.*
Foliation complexity

Let $\mathcal{F}$ be a foliation of a compact Riemannian manifold $M$.

For each $w \in M$ the leaf $L_w$ containing $w$ inherits a Riemannian metric for which $L_w$ is geodesically complete.

Fix $L_w$ and then count the number of points $Gr(L_w, d) = \#\{L_w \cap T\}$.

The rate of growth of the function $d \mapsto Gr(L_w, d)$ is a measure of the complexity of the leaf.

$L_w$ has exponential growth type if there exists $\lambda > 0$ and $d_0 \geq 0$ such that

$$Gr(L_w, d) \geq \exp\{\lambda \cdot d\}, \quad d \geq d_0$$

$L_w$ has polynomial growth type if there exists $m > 0$ and $d_0 \geq 0$ such that

$$Gr(L_w, d) \leq m^k, \quad d \geq d_0$$
Subexponential complexity

**Definition:** A foliation $\mathcal{F}$ is *uniformly subexponential* if:

for all $\epsilon > 0$, there exists $C_\epsilon, d_\epsilon$ so that for all $w \in M$,

$$Gr(L_w, d) \leq C_\epsilon \cdot \exp\{\epsilon \cdot d\} \quad , \quad d \geq d_\epsilon$$

The celebrated theorem of Connes, Feldman and Weiss [1981] implies:

**Theorem:** If $\mathcal{F}$ is uniformly subexponential, then the equivalence relation it defines on the transversal space $\mathcal{T}$ is amenable, hence hyperfinite.

The pseudogroup action $G_\mathcal{F}$ on $\mathcal{T}$ is *hyperfinite* if it is *measurably orbit equivalent* to an action of the integers $\mathbb{Z}$ on the interval $[0, 1]$.

**Moral:** Subexponential complexity often leads to ambiguity.
Expansion growth

We measure exponential complexity for pseudogroup actions, following [Bowen 1971] and [Ghys, Langevin & Walczak 1988].

Let $\epsilon > 0$ and $d > 0$. A subset $\mathcal{E} \subset \mathcal{T}$ is said to be $(\epsilon, d)$-separated if

- for all $w, w' \in \mathcal{E} \cap \mathcal{T}_i$
- there exists $g \in \mathcal{G}_\mathcal{F}$ with $w, w' \in \text{Dom}(g) \subset \mathcal{T}_i$, and $\|g\| \leq d$
- then $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$.
- If $w \in \mathcal{T}_i$ and $w' \in \mathcal{T}_j$ for $i \neq j$ then they are $(\epsilon, d)$-separated by default.

The “expansion growth function” counts the maximum of this quantity:

$$h(\mathcal{G}_\mathcal{F}, \epsilon, d) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathcal{T} \text{ is } (\epsilon, d)\text{-separated}\}$$

If the pseudogroup consists of isometries, for example, then applying elements of $\mathcal{G}_\mathcal{F}$ does not help to separate points, so these growth functions remain polynomial as functions of $d$, for all $\epsilon$. 
Foliation geometric entropy

The function \( d \mapsto h(\mathcal{G}_\mathcal{F}, \epsilon, d) \) measures expansion growth at distance \( d \) - sort of an integrated total exponent.

Define:

\[
h(\mathcal{G}_\mathcal{F}, \epsilon) = \limsup_{d \to \infty} \frac{\ln \left\{ \max \{ \# \mathcal{E} \mid \mathcal{E} \text{ is } (\epsilon, d)-\text{separated} \} \right\}}{d}
\]

\[
h(\mathcal{G}_\mathcal{F}) = \lim_{\epsilon \to 0} h(\mathcal{G}_\mathcal{F}, \epsilon)
\]

**Theorem:** [GLW 1988] The quantity \( h(\mathcal{G}_\mathcal{F}) \) is finite if \( \mathcal{F} \) is a \( C^1 \)-foliation. Moreover, the property \( h(\mathcal{G}_\mathcal{F}) > 0 \) is independent of all choices.

**Theorem:** If \( \mathcal{F} \) is defined by a flow \( \phi_t \) then \( h(\mathcal{G}_\mathcal{F}) = 2 \cdot h_{top}(\phi_1) \).
Doubling maps have entropy $\ln(2) > 0$

**Exercise:** The Hirsch foliations always have positive geometric entropy.

**Solution:** The holonomy pseudogroup $\mathcal{G}_F$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After $d$-iterations, the inverse map to $z \mapsto z^{2^d}$ has derivative of norm $2^d$ so we have a rough estimate

$$h(\mathcal{G}_F, \epsilon, d) \sim \left(\frac{2\pi}{\epsilon}\right) \cdot 2^d$$
Orbit growth implies entropy

For the Hirsch example, notice as we wander out the tree-like leaf, we are also wandering around the transversal space $\mathcal{T}$. 
Manning’s Theorem

Let $B$ be a compact manifold of non-positive curvature.
Let $M = T^1 B$ denote the unit tangent bundle to $B$.
Let $\phi_t : M \to M$ be the geodesic flow of $B$.

**Theorem:** [Manning 1976] $h_{top}(\phi) = Gr(\pi_1(B, b_0))$

That is, the growth rate of the volume of balls in the universal covering of $B$ equals the entropy.

This is actually a theorem about foliation entropy and growth rates of leaves.
Fundamental domains

The assumption that $B$ has non-positive curvature implies that its universal covering $\tilde{B}$ is a disk, and we can “color” it with fundamental domains:

The proof of Manning’s Theorem follows from the picture.
Assume that $B$ has uniformly negative sectional curvatures.

Let $\phi_t: M \to M$ be the geodesic flow. Define an equivalence relation on points of $M$:

$$w \sim_\phi w' \iff d_M(\phi_t(w), \phi_t(w')) \leq C \text{ for } t \to \infty$$

Then define

$$L_w = \{ w' \in M \mid w' \sim_\phi w \}$$

**Theorem:** [Pugh-Shub 1974] The sets $L_w$ form the leaves of a $C^1$-foliation of $M$. The resulting foliation is called the *weak-stable foliation* for $\phi_t$.

1) Each leaf $L_w$ is a $C^\infty$-immersed submanifold of $M$.

2) The orbits of the geodesic flow $\phi_t(w)$ are contained in the leaves of $\mathcal{F}$.
**Theorem:** Let $B$ be a compact manifold of negative curvature, and let $\mathcal{F}$ be the weak stable foliation for the geodesic flow $\phi_t$. Then

$$h(\mathcal{G}_\mathcal{F}) = 2 \cdot h_{\text{top}}(\phi_1)$$

The proof that $h(\mathcal{G}_\mathcal{F}) \geq 2 \cdot h_{\text{top}}(\phi_1)$ is easy - we use the holonomy along geodesic segments to separate points.

The other estimate requires knowing about the structure of the weak stable foliations - the leaves are obtained by applying the geodesic flow to the strong stable foliations, which are polynomial growth, so do not add any exponential complexity.
Entropy and chaos

**Question:** When is \( h(G_F) > 0? \)

- Expanding holonomy (Hirsch examples)
- Weak stable foliations (for Anosov flows)
- Ping-pong games (Resilient leaves in codimension one)

Are there other canonical situations where we can expect positive entropy?

For example, if \( F \) has leaves of exponential growth, does there always exist a \( C^1 \)-close perturbation of \( F \) with positive entropy?

Next time, we discuss the relation between foliation entropy and the existence of hyperbolic invariant measures for the foliation geodesic flow.
Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures $\mu_*$ for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]:

Friday [7/5/2010]:


