Lecture 4: Entropy and Exponent

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Problem 2: What hypotheses on the dynamics of $\mathcal{F}$ are sufficient to imply that $h(\mathcal{G}_{\mathcal{F}}) > 0$?

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Problem 3: Are there cohomology hypotheses on $\mathcal{F}$ which would "improve" our understanding of its dynamics? How does leafwise cohomology $H^*(\mathcal{F})$ influence dynamics? Secondary invariants for $\mathcal{F}$?

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The following result has various applications, especially in codimension one. Best news – it introduces a new technique.

Theorem: [G-L-W 1998] Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension $q \geq 1$. If $h(\mathcal{G}_{\mathcal{F}}) = 0$, then the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ admits an invariant probability measure.

Three Theorems

Theorem: [H 2000] Let M be compact with a C^r-foliation $\mathcal{F}$ of codimension-q. If $q = 1$ and $r \geq 1$, or $q \geq 2$ and $r > 1$, then

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Theorem: [H & Langevin 2000] Let M be compact with a codimension one, C^2-foliation $\mathcal{F}$. Then

$$0 \neq GV(\mathcal{F}) \in H^3(M, \mathbb{R}) \quad \Longrightarrow \quad h(\mathcal{G}_{\mathcal{F}}) > 0$$

Positive exponents

Proposition: Let $\mathcal{F}$ be C^1, and suppose $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there is a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow.

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The proof illustrates the tools available.

Assume $h(\mathcal{G}_{\mathcal{F}}) = 2\lambda$. Then for sufficiently small $\epsilon > 0$, for $d \gg 0$ there exists $\mathcal{E} = \{w_1, \ldots, w_\ell\}$ where $\ell > \exp\{d \cdot \lambda\}$, so that for $w_i \neq w_j$ there exists some path $\tau_{i,j} \colon [0,1] \to L_{w_i}$ with

$$d_{\mathcal{T}}(h_{\tau_{i,j}}(w_i), h_{\tau_{i,j}}(w_j)) \geq \epsilon$$

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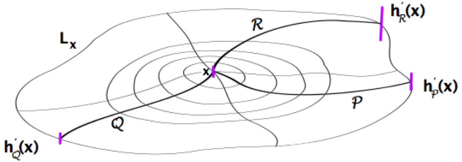
Mean Value Theorem $\Longrightarrow h'_{\tau_{i,j}}(w'_i) > \exp\{d \cdot \lambda\}$

Quivers

Actually, Pigeon Hole Principle implies there are closed neighborhoods $D(w, \delta) \subset \mathcal{T}$ containing an exponential number of such points:



These are call *quivers*.

Hyperbolic measures

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Proposition: Let $\mathcal{F}$ be C^1 and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists an exponential collection of leafwise geodesic segments $\{\gamma_\ell \colon [0, \infty) \to M$ along which the normal derivative cocycle Dh_{γ_ℓ} has exponentially decreasing directions.

Stable manifolds

Theorem: Let $\mathcal{F}$ be $C^{1+\alpha}$ and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists an exponential collection of leafwise geodesic segments $\{\gamma_\ell \colon [0, \infty) \to M$ with stable transverse manifolds.

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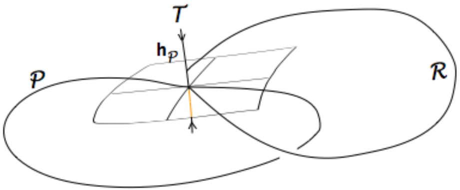
Corollary: Let $\mathcal{F}$ be a codimension one, C^1-foliation, or codimension $q > 1$ $C^{1+\alpha}$-foliation. Then $\mathcal{G}_{\mathcal{F}}$ distal implies that $h(\mathcal{G}_{\mathcal{F}}) = 0$.

Theorem: Let $\mathcal{F}$ be C^1 and suppose that $h(\mathcal{G}_\mathcal{F}) > 0$. Then $\mathcal{G}_\mathcal{F}$ acting on $\mathcal{T}$ admits a "ping-pong game" which implies the existence of a resilient leaf for $\mathcal{F}$.



Problemos de la diá

Monday [3/5/2010]: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday [4/5/2010]: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday [5/5/2010]: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday [6/5/2010]: Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which implies positive entropy.

Friday [7/5/2010]: Characterize the exceptional minimal sets of zero entropy.

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