Critical point theory for foliations

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder

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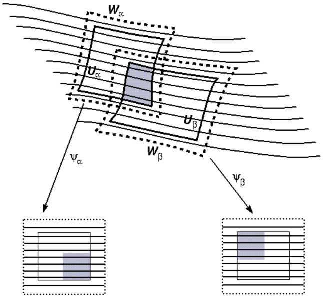
- with respect to a foliation $\mathcal{F}$ of M, where we require that f be a foliated map.

Definition: $f\colon M \to M'$ is a foliated map, if $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ are foliated spaces and f maps leaves of F to leaves of $\mathcal{F}'$.

Last example is the most general, and includes the previous cases.

Definition of foliation:

A foliation $\mathcal{F}$ of dimension p on a manifold M is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



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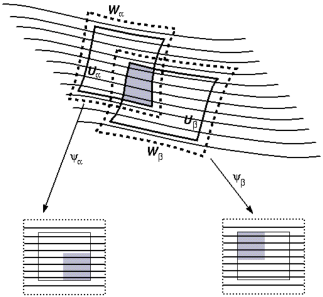

If the dimensions of the leaves are not constant, then we say $\mathcal{F}$ is a *singular foliation*. Many of the results of this talk apply also to singular foliations.

Foliations arise in the study of many subjects:

1. Partial differential equations (Reeb, Godbillon, Sacksteder)
2. Representation theory - cocycles, co-orbit spaces, W^* & C^*-algebras (Murray – von Neumann, Mackey, Kasparov)
3. Generalized dynamical systems (Anosov, Smale, Hirsch, Shub, Hector)
4. Topology of classifying spaces (Bott, Haefliger, Gelfand-Fuks, Mather, Morita, Thurston)
5. Geometry - open book decompositions and laminations of manifolds (Lawson, Winkelnkemper, Thurston, Gabai)
6. Physics & Non-Commutative Geometry (Bellisard, Connes, et al)

Examples: compact foliations

We say that $\mathcal{F}$ is a *compact foliation* if every leaf L of $\mathcal{F}$ is a compact submanifold. $\mathcal{F}$ is a *compact Hausdorff foliation* if every leaf is compact and the quotient space $M' = M/\mathcal{F}$ is Hausdorff. Here are three examples:

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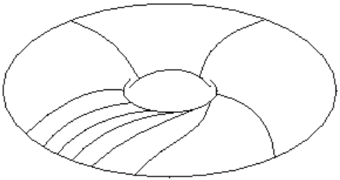
- Let M^3 be a Seifert fibered 3-manifold, fibered by circles, with base space B an orbifold.
- G a compact connected Lie group and $G \times M \to M$ locally free.

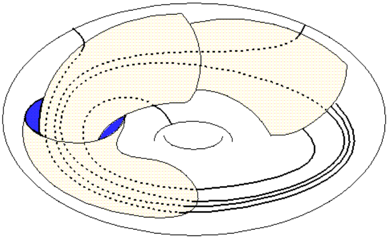
Not every compact foliation is compact Hausdorff, so even in this simplest class of foliations, their study is complicated.

Examples: non-commutative tori



Lines fill up the 2-torus $\mathbb{T}^2$

Examples: Reeb foliation of $\mathbb{S}^3$



Planes fill up the solid 2-torus.
Two copies of the torus glue together to give $\mathbb{S}^3$.

Examples: Lie group actions

Let G be a connected Lie group and $\mathbf{K}$ a compact topological space, with a continuous action $\varphi\colon G \times \mathbf{K} \to \mathbf{K}$. If all orbits of φ have the same dimension, then the action defines a *lamination* of $\mathbf{K}$.

Examples include:

- Locally free action of a Lie group on a compact manifold M
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If the orbits of G have varying dimensions, then we get a singular foliation.

Examples: discrete group actions

Let Γ be a finitely-generated group and N a compact manifold of dimension q, with a smooth action $\alpha\colon \Gamma \times N \to N$.

Then there exists a compact $q+2$-dimensional manifold M with foliation $\mathcal{F}_\alpha$ having 2-dimensional leaves, such that the global holonomy of $\mathcal{F}_\alpha$ is conjugate to the representation α.

The point is that the geometry (more precisely, the holonomy) of $\mathcal{F}$ captures all of the information about the given group action.

(The construction of $\mathcal{F}$ uses a sequence of "twisted surgeries" on $\mathbb{S}^2 \times N$, one for each generator of Γ.)

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Lemma: If $f : M \to \mathbb{R}$ is a foliated map for the point foliation $\mathcal{F}'$ on $\mathbb{R}$, then for all $c \in \mathbb{R}$ the inverse image $X_c = f^{-1}(c)$ is a closed, saturated subset.

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Corollary: Assume that $f\colon M \to \mathbb{R}$ is proper, then for each critical value $c \in \mathbb{R}$ of f, there exists a minimal set $\mathbf{K}_c \subset X_c$ with $f(\mathbf{K}_c) = c$.

Counting critical points

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The classical case: let $f : M \to \mathbb{R}$ be a C^1-function on a closed Riemannian manifold M.

Theorem: (Lusternik-Schnirelmann [1934])

$$\#\{x \mid x \in M \text{ is critical for } f\} \geq \mathrm{cat}(M)$$

where $\mathrm{cat}(M)$ is the Lusternik-Schnirelmann category of M, which is defined as the least number of *open* sets $\{U_1, \ldots, U_k\}$ required to cover M such that each U_ℓ is contractible in M to a point.

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Colman defined transverse LS-category for foliations in her Thesis [1998].

Transverse LS category of foliations

Let $(M, \mathcal{F})$ be a foliated manifold, and $U \subset M$ and open saturated subset.

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Definition: (Colman) The transverse LS category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of transversely categorical open saturated sets required to cover M. If no such covering exists, then set $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

Transverse LS category of foliations – examples

Example: Let $M \to M'$ be a fibration with compact fibers which defines the foliation $\mathcal{F}$ on M. Then $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}(M')$.

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Theorem: (Colman [1998]) If $\mathcal{F}$ is compact Hausdorff, then $\mathrm{cat}_{\pitchfork}(M,\mathcal{F})$ is finite. Moreover, the Lusternik-Schnirelmann estimate holds for counting the number of critical leaves:

$$\#\{L \mid L \subset M \text{ is critical for } f\} \geq \mathrm{cat}_{\pitchfork}(M,\mathcal{F})$$

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In both examples above, we see the problem arises from the properties of the leaf closures of $\mathcal{F}$.

Theorem: (Hurder [2000]) Let $k = \mathrm{cat}_{\pitchfork}(M, \mathcal{F}) < \infty$. Given a transversally categorical covering of M, $\{H_{\ell,t} \colon U_\ell \to M \mid 1 \leq \ell \leq k\}$ with $H_{\ell,1}(U_\ell) \subset L_\ell$, then each L_ℓ is a compact leaf.

Corollary: $\mathcal{F}$ has no compact leaves $\Rightarrow \mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$.

Counting critical minimal sets

The basic observation is that compact leaves are just a special case of compact minimal sets.

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Solution: Modify the definition of transversally categorical set.

Essential transverse LS category of foliations

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Definition: The essential transverse LS category $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$ of a foliated manifold $(M, \mathcal{F})$ is the least number of essentially transversely categorical open saturated sets required to cover M. If no such covering exists, then set $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \infty$.

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Remark: $\mathcal{F}$ a compact foliation $\Longrightarrow \mathrm{cat}^e_{\pitchfork}(M, \mathcal{F}) = \mathrm{cat}_{\pitchfork}(M, \mathcal{F})$.

Riemannian foliations

Definition: $\mathcal{F}$ is a Riemannian foliation if there is a Riemannian metric on TM so that the restriction to the normal bundle $Q = T\mathcal{F}^{\perp}$ is invariant under the leafwise parallelism.

Equivalently, the induced metric on Q is locally projectable: for any open set $U \subset M$ such that $\mathcal{F} \mid U$ is defined by a fibration $\pi_U \colon U \to B_U$ then the map π_U is a local Riemannian submersion.

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- $\mathcal{F}$ defined by locally free action of compact connected Lie group $\Longrightarrow$ $\mathcal{F}$ is Riemannian.
- $\mathcal{F}$ compact Hausdorff $\Longrightarrow$ $\mathcal{F}$ is Riemannian.
- For each leaf L, the closure $\overline{L}$ is a minimal set of $\mathcal{F}$.

Main Theorem

Theorem: (Hurder-Töben [2006]) Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M. Then:

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- Let $f \colon M \to \mathbb{R}$ be a foliated map. Then we have a generalized form of the Lusternik-Schnirelmann Theorem:
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The proof actually gives much more information.

Molino Theory

Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M.

Let $\widehat{M}$ denote the bundle of orthonormal frames to $\mathcal{F}$ – $\pi \colon \widehat{M} \to M$ is an $\mathbf{O}(q)$-fibration with a right action of $\mathbf{O}(q)$.

Theorem: (Molino [1982])

- The foliation $\mathcal{F}$ "lifts" to a Riemannian foliation $\widehat{\mathcal{F}}$ of $\widehat{M}$ whose leaves cover those of $\mathcal{F}$
- $\widehat{\mathcal{F}}$ is $\mathbf{O}(q)$-equivariant.
- For each leaf $\widehat{L}$ of $\widehat{\mathcal{F}}$, the closure $\overline{\widehat{L}}$ is a submanifold of $\widehat{M}$ (and a minimal set for $\widehat{\mathcal{F}}$.)
- The closures of the leaves of $\widehat{\mathcal{F}}$ form a compact foliation $\overline{\widehat{\mathcal{F}}}$ of $\widehat{M}$
- The leaf space $\widehat{W} = \widehat{M}/\overline{\widehat{\mathcal{F}}}$ is a manifold, and the quotient map $\widehat{\Upsilon} \colon \widehat{M} \to \widehat{W}$ is an $\mathbf{O}(q)$-equivariant Riemannian submersion.

Equivariant foliated LS category

A foliated C^r-map $f\colon M \to \mathbb{R}$ induces an $\mathbf{O}(q)$-invariant map $\widehat{f}\colon \widehat{W} \to \mathbb{R}$.

Proposition: Critical minimal sets of $f \iff$ critical orbits of $\widehat{f}$

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Theorem: (Hurder-Töben [2006]) Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M. Then $\mathrm{cat}^e_{\pitchfork}(M,\mathcal{F}) = \mathrm{cat}_{\mathbf{O}(q)}(\widehat{W})$.

Equivariant foliated LS category

A foliated C^r-map $f\colon M \to \mathbb{R}$ induces an $\mathbf{O}(q)$-invariant map $\widehat{f}\colon \widehat{W} \to \mathbb{R}$.

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Corollary: Let $f\colon M \to \mathbb{R}$ be a foliated map.

$$\#\{\mathbf{K}_c \mid \mathbf{K}_c \subset M \text{ is critical for } f\} \geq \mathrm{cat}_{\mathbf{O}(q)}(\widehat{W})$$

Hence, one can use the full-force of equivariant LS category theory to calculate $\mathrm{cat}^e_{\pitchfork}(M, \mathcal{F})$ and estimate the number of critical minimal sets.

Polar actions

Definition: Let G Lie group acting smoothly by isometries on a complete Riemannian manifold M. A *section* for the G-action is an isometrically immersed complete submanifold $i : \Sigma \longrightarrow M$ which meets every orbit and always orthogonally.

The dimension of Σ is equal to the cohomogeneity of the action, denoted by q. Note that for any $g \in G$, the map $g \circ i \colon g\Sigma \longrightarrow M$ is again a section.

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Definition: A *polar action* is a G-action with a section. If Σ is a flat submanifold, then the action is called *hyperpolar*.

The geometry of polar actions has been extensively studied by Kostant [1973], Szenthe [1984], Dadok [1985], Palais & Terng [1988], Thorbergsson [1999], Kollross [2002], and others.

Examples of polar actions

1. Isometric cohomogeneity one actions. The sections are the normal geodesics of a regular orbit. These have been classified in special cases by Kollross and Berndt & Tamaru, although remains an open problem to classify all such actions.

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3. Let N be a symmetric space. The identity component of the isometry group, $G = I(N)_0$, acts transitively on N. We can write $N = G/K$, where $K = G_p$ for some point $p \in N$, and (G, K) is called a *symmetric pair*. Then the isotropy action

$$K \times G/K \to G/K \ ; \quad (k, gK) \mapsto kgK$$

and its linearization $K \times (T_{[K]}G/K) \to T_{[K]}G/K$ at the the tangent space to the point [K], are hyperpolar. The sections are the maximal flat submanifolds through $[K]$, and their tangent spaces in $[K]$, respectively. These are called *s-representations*.

Weyl group

Let G a Lie group acting smoothly by isometries on a complete Riemannian manifold M, and assume the action is polar with section $i : \Sigma \to M$. Let

$$\begin{aligned} N := N_G(\Sigma) &= \{g \in G \mid g(i(\Sigma)) = i(\Sigma)\} \\ Z := Z_G(\Sigma) &= \{g \in G \mid gi(x) = i(x) \text{ for any } x \in \Sigma\} \end{aligned}$$

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Definition: The Weyl group is

$$W_G(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$$

In the case of Example (2) above, this is just the usual Weyl group.

Category for polar actions

Theorem: (Hurder-Töben [2007]) Let G be a Lie group with a proper polar action on M, $i : \Sigma \to M$ a section, and $W = N_G(\Sigma)/Z_G(\Sigma)$ the generalized Weyl group acting on Σ. Then

$$\mathrm{cat}_G(M) \leq \mathrm{cat}_W(\Sigma) \tag{1}$$

Proof uses ideas and techniques developed for the study of the transverse LS category of Riemannian foliations (especially the lifting of foliated homotopies via the Ehresmann connection on Riemannian submersions.)

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As an application, we obtain a well-known result of Wilhelm Singhoff:

Theorem: (Singhoff [1975]) The LS-categories of the unitary and the special unitary groups are $\mathrm{cat}(SU(n)) = n$ and $\mathrm{cat}(U(n)) = n + 1$.

Some open problems / works in progress

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3. Classify the Riemannian foliations for which the action $\mathbf{O}(q)$ on $\widehat{W}$ is polar, or hyperpolar.
4. Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold. Relate $\mathrm{cat}^{e}_{\pitchfork}(M, \mathcal{F})$ to the transverse Euler characteristic (Hopf index) of $\mathcal{F}$.