Dynamics of Derived from Kuperberg flows

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joint work with Ana Rechtman

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At the *International Symposium/Workshop on Geometric Study of Foliations* in November 1993, there was an evening seminar on a recent preprint by Krystyna Kuperberg, proving:

Theorem (K. Kuperberg, 1994) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

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- Krystyna Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.
- Étienne Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, 227: 283–307, 1995.
- Shigenori Matsumoto, *K.M. Kuperberg's C^∞ counterexample to the Seifert conjecture*, **Sūgaku**, Mathematical Society of Japan, Vol. 47:38–45, 1995. Translation: **Sugaku Expositions**, A.M.S., Vol. 11:39–49, 1998.
- Greg & Krystyna Kuperberg, *Generalized counterexamples to the Seifert conjecture*, **Ann. of Math. (2)**, 144:239–268, 1996.

These papers also studied the dynamics of Kuperberg flows:

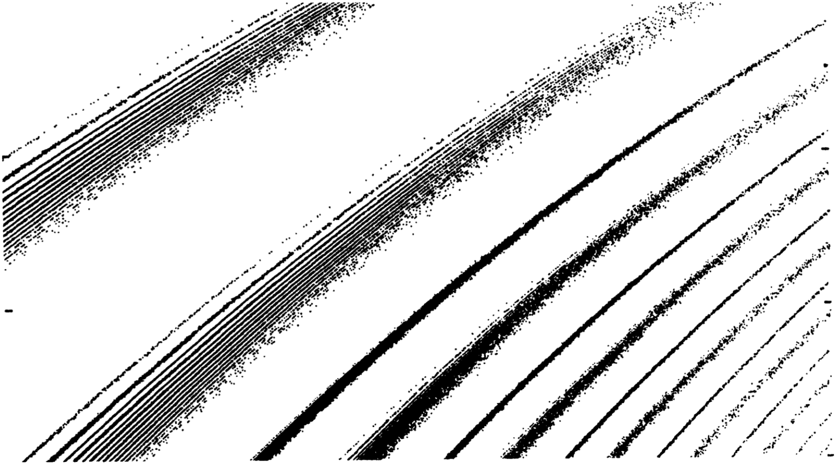
Theorem (Ghys, Matsumoto, 1995) *The Kuperberg flow has a unique minimal set* $\mathcal{Z} \subset M$.

Theorem (Matsumoto, 1995) *The Kuperberg flow has an open set* $\mathfrak{W}$ *of wandering points whose forward orbits limit to the unique minimal set.*

There cannot be an invariant measure equivalent to Lebesgue.

Theorem (A. Katok, 1980) *Let* M *be a closed, orientable 3-manifold. Then an aperiodic flow* ϕ_t *on* M *has zero entropy.*

The other "clue" to the dynamics of Kuperberg flows, was given by the computer model of Bruno Sévennec of the intersection of the unique minimal set with a cross section. Very mysterious!



Problem: [Kuperberg] Describe the topological shape of the unique minimal set $\mathcal{Z}$. Does $\mathcal{Z}$ have *stable shape*?

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- H. & R., *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377, 2016, 250 pages.
- H. & R., *Aperiodic flows at the boundary of chaos, arXiv:1603.07877*.
- H. & R., *Perspectives on Kuperberg flows, arXiv:1607.00731*.
- H. & R., *The dynamics of Derived from Kuperberg flows, in preparation*.

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Let Φ_t be a generic Kuperberg flow on a plug $\mathbb{K}$.

Theorem (H & R, 2015) *The minimal set $\mathfrak{M}$ is a 2-dimensional lamination "with boundary", equal to the non-wandering set.*

Theorem (H & R, 2015) *The minimal set $\mathfrak{M}$ has unstable shape.*

This implies that it is wildly embedded in $\mathbb{R}^3$.

Theorem (H & R, 2015) *The flow Φ_t has positive "slow entropy", for exponent $\alpha = 1/2$.*

Thus, a generic Kuperberg flow *almost* has positive entropy.

We also considered "Derived from Kuperberg" constructions, which deform a generic Kuperberg flow $\Phi_t = \Phi_t^0$ on a plug $\mathbb{K}$.

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $-1 < \epsilon \leq 0$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ has two periodic orbits, and all orbits are properly embedded.*

Theorem (H & R, 2016) *There is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $0 \leq \epsilon < a$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ admits countably many families of "horseshoes" with dense periodic orbits, and so has positive entropy.*

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Conclusion 1: *The generic Kuperberg flows lie at the boundary of chaos (entropy > 0) and the boundary of tame dynamics.*

Conclusion 2: *There is still much about the dynamics of Derived from Kuperberg flows that we still don't understand.*

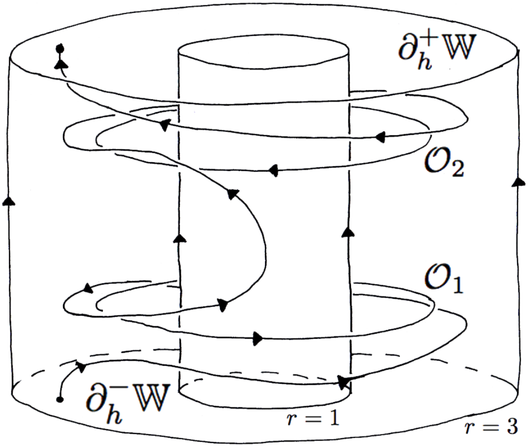
Definition: A plug is a 3-manifold with boundary of the form $P = D \times [-1, 1]$ with D a compact surface with boundary. P is endowed with a non-vanishing vector field $\vec{X}$, such that:

- $\vec{X}$ is vertical in a neighborhood of ∂P, that is $\vec{X} = \frac{d}{dz}$. Thus $\vec{X}$ is inward transverse along $D \times \{-1\}$ and outward transverse along $D \times \{1\}$, and parallel to the rest of ∂P.
- There is at least one point $p \in D \times \{-1\}$ whose positive orbit is trapped in P.
- If the orbit of $q \in D \times \{-1\}$ is not trapped then its orbit intersects $D \times \{1\}$ in the facing point.
- There is an embedding of P into $\mathbb{R}^3$ preserving the vertical direction.

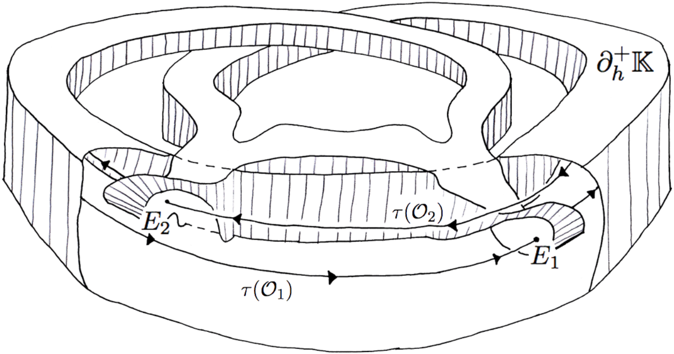
Modified Wilson Plug $\mathbb{W}$

Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f\,\frac{f}{d\theta}$

f is asymmetric in z and $\vec{W}_1 = g\,\frac{f}{dz}$ is vertical.



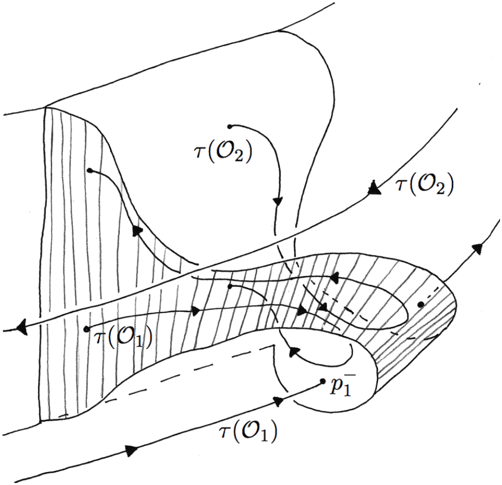
Grow horns and embed them to obtain Kuperberg Plug $\mathbb{K}$, matching the flow lines on the boundaries.



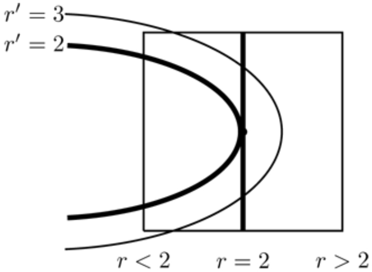
Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself.

That's it. The resulting flow is aperiodic!

Close up view of the lower embedding σ_1



The insertion map as it appears in the face E_1



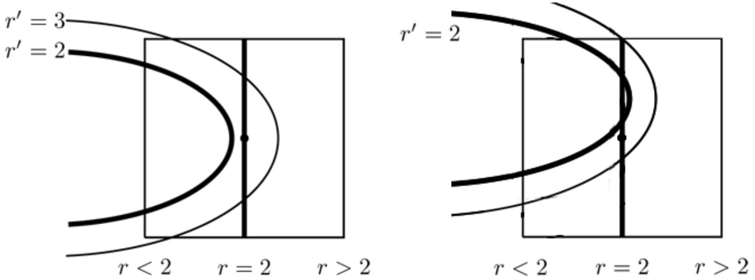
Radius Inequality:

For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r'$ unless $x' = (2, \theta_i, -2)$ and then $r = 2$.

Definition: A *Derived from Kuperberg* (DK) flow is obtained by choosing the embeddings so that we have:

Parametrized Radius Inequality: For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

The modified radius inequality for the cases $\epsilon < 0$ and $\epsilon > 0$:



Proposition: *Let Φ_t^ϵ be a DK flow for which the insertion map satisfies the Parametrized Radius Inequality with $\epsilon < 0$. Then the flow in the plug $\mathbb{K}_\epsilon$ has two periodic orbits that bound an invariant cylinder, and the flow has topological entropy zero.*

Idea of the proof: This follows from the techniques for the standard flow when $\epsilon = 0$, which imply that every flow orbit of a point x with radius $r(x) \neq 2$ entering an insertion, exits at the same radius.

There are two types of *generic* conditions.

Generic 1: Consider the rectangle $\mathbf{R} = [1,3] \times [-2,2]$, with a vertical vector field $\vec{W}_1 = g\frac{f}{dz}$ where $g(r,z)$ vanishes at $(2,-1)$ and $(2,+1)$. We require that g *vanish to second order with positive definite Jacobian* at these two points.

Then the Wilson field on $\mathbb{W} = \mathbf{R} \times \mathbb{S}^1$ is $\vec{W} = \vec{W}_1 + f\frac{f}{d\theta}$ where f is asymmetric in z, and vanishes near the boundary.

Generic 2: These are conditions on the insertion maps $\sigma_i \colon D_i \to \mathcal{D}_i$. We require that the r-coordinate of the image *depends quadratically* on the θ-coordinate of the domain, for values of r near $r = 2$. Much stronger than the basic radius inequality.

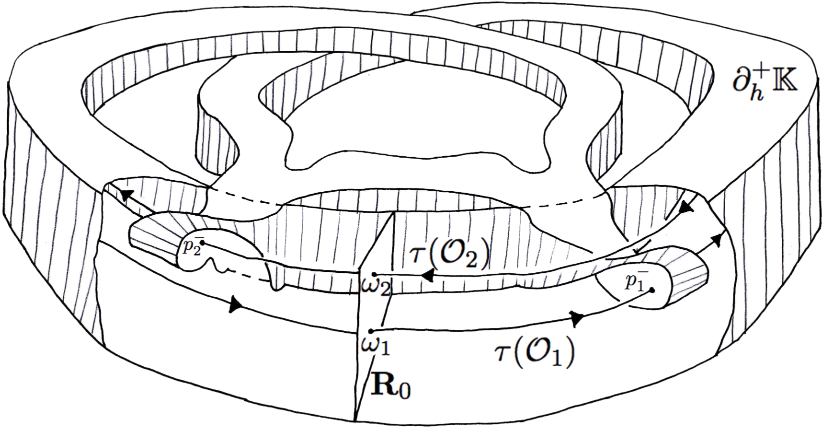
Problem: *What are the dynamical properties of the non-generic Kuperberg flows? Shape of its minimal set?*

The basic idea for the study of the minimal set for a Kuperberg flow, and the dynamics of a Derived from Kuperberg flow Φ_t^ϵ where $\epsilon > 0$, is to compare the dynamics of the flow Φ_t^0 with that of an induced map on a (partial) section to the flow.

Return map of a flow Φ_t^ϵ induces a smooth pseudogroup $\mathcal{G}_{\Phi^\epsilon}$ on $\mathbf{R}_0$

Critical difficulty: *There is not always a direct relation between the continuous dynamics of the flow Φ_t^ϵ and the discrete dynamics of the action of the pseudogroup $\mathcal{G}_{\Phi^\epsilon}$.*

The section $\mathbf{R}_0 \subset \mathbb{K}$ used to define pseudogroup $\mathcal{G}_{\Phi^\epsilon}$.



The flow of Φ_t^ϵ is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$, so the action of the pseudogroup has singularities along this line.

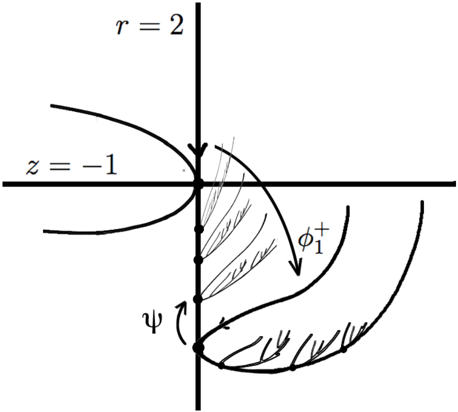
We consider two maps with domain in $\mathbf{R}_0$

- ψ which is the return map of the *Wilson flow* Ψ_t
- ϕ_1^ϵ which is the return map of the *Kuperberg flow* Φ_t^ϵ for orbits that go through the entry region E_1

Form the pseudogroup they generate $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$.

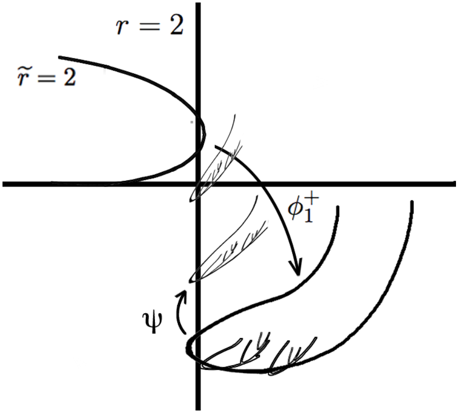
Proposition: The restriction of $\widehat{\mathcal{G}}_\epsilon$ to the region $\{r > 2\} \cap \mathbf{R}_0$ is a sub-pseudogroup of $\mathcal{G}_{\Phi^\epsilon}$

Action of $\widehat{\mathcal{G}}_0 = \langle \psi, \phi_1^{\epsilon} \rangle$ on the line $r = 2$ for $\epsilon = 0$.



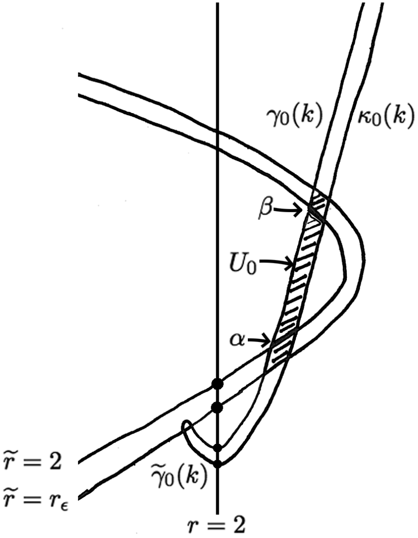
This looks like a ping-pong game, except that the play action is too slow to generate entropy.

Action of $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$ on the line $r = 2$ for $\epsilon > 0$.

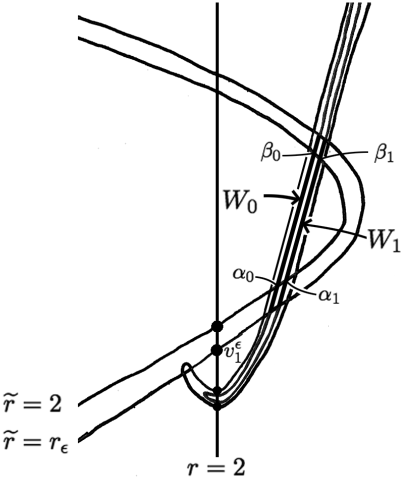


The dynamics of this action is actually too complicated to draw precisely, or calculate with.

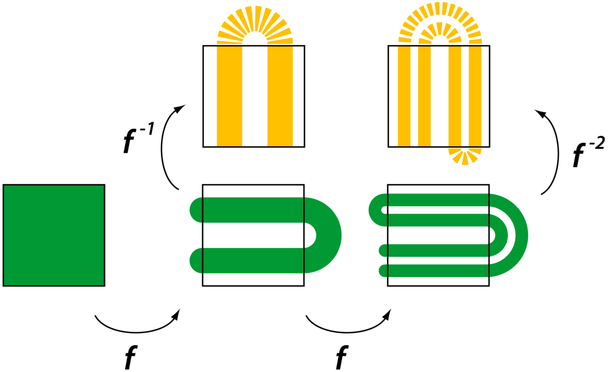
Define a compact region $U_0 \subset \mathbf{R}_0$ which is mapped to itself by the map $\varphi = \psi^k \circ \phi_1^\epsilon$ for k sufficiently large.



The images of the powers φ^ℓ of the map the map φ form a δ-separated set for the action of the pseudogroup $\widehat{\mathcal{G}}_\epsilon$.



This is a horseshoe dynamical system for the *pseudogroup action*:



Definition: Let $\mathfrak{Z} \subset X$ be a continuum embedded in a metric space X. A *shape approximation* of $\mathfrak{Z}$ is a sequence $\mathfrak{U} = \{U_\ell \mid \ell = 1, 2, \ldots\}$ satisfying the conditions:

1. each U_ℓ is an open neighborhood of $\mathfrak{Z}$ in X which is homotopy equivalent to a compact polyhedron;
2. $U_{\ell+1} \subset U_\ell$ for $\ell \geq 1$, and their closures satisfy $\bigcap_{\ell \geq 1} \overline{U}_\ell = \mathfrak{Z}$.

Shape homotopy groups:

$$\widehat{\pi}_1(\mathcal{Z}, x_*) \equiv \varprojlim \; \{\pi_1(U_{\ell+1}, x_*) \to \pi_1(U_\ell, x_*)\}$$

Meta-Principle: For each class $[\gamma] \in \widehat{\pi}_1(\mathfrak{M}, \omega_0)$ there is a horseshoe subdynamics for the pseudogroup $\mathcal{G}_{\Phi^\epsilon}$ acting on $\mathbf{R}_0$ with a periodic orbit defining $[\gamma]$.

Conclude with two remarks:

- For $\epsilon < 0$, the dynamics of a DK flow is tame, and completely predictable, except that as $\epsilon \to 0$ the dynamics approaches that of the Kuperberg flow.

- For $\epsilon > 0$, the dynamics of a DK flow is chaotic, but making calculations of entropy for example, is only possible in special instances. Also, we have no intuition, for example, of how to describe the nonwandering sets for DK flow with $\epsilon > 0$.

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Thank you for your attention!