Homogeneous matchbox manifolds

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Groupoidfest 2009, October 24, 2009
Continua...

**Definition:** A *continuum* is a compact and connected metrizable space.

**Definition:** An *indecomposable continuum* is a continuum that is not the union of two proper subcontinua.

**Examples:** The circle $S^1$ is decomposable. The Knaster Continuum (or *bucket handle*) is indecomposable.

This is one-half of a Smale Horseshoe. The 2-solenoid over $S^1$ is a branched double-covering of it.
Indecomposable continuum arise naturally as invariant closed sets of dynamical systems; e.g., attractors and minimal sets for diffeomorphisms.

Definition: A space $X$ is homogeneous if for every $x, y \in X$ there exists a homeomorphism $h: X \to X$ such that $h(x) = y$. Equivalently, $X$ is homogeneous if the group $\text{Homeo}(X)$ acts transitively on $X$.

Question: [Bing1960] If $X$ is a homogeneous continuum and if every proper subcontinuum of $X$ is an arc, must $X$ then be a circle or a solenoid?

Theorem: [Hagopian 1977] Let $X$ be a homogeneous continuum such that every proper subcontinuum of $X$ is an arc, then $X$ is an inverse limit over the circle $S^1$. 
**Question:** Let $X$ be a homogeneous continuum such that every proper subcontinuum of $X$ is an $n$-dimensional manifold, must $X$ then be an inverse limit of normal coverings of compact manifolds?

We rephrase the context:

**Definition:** An $n$-dimensional *matchbox manifold* is a continuum $M$ which is a foliated space with leaf dimension $n$, and codimension zero.

$M$ is a foliated space if it admits a covering $\mathcal{U} = \{\varphi_i \mid 1 \leq i \leq \nu\}$ with foliated coordinate charts $\varphi_i : U_i \to [-1, 1]^n \times \mathcal{T}_i$. The compact metric spaces $\mathcal{T}_i$ are totally disconnected $\iff M$ is a matchbox manifold.

The leaves of $\mathcal{F}$ are the path components of $M$. 
Smooth matchbox manifolds

**Definition:** $\mathcal{M}$ is a *smooth foliated space* if the leafwise transition functions for the foliation charts $\varphi_i : U_i \rightarrow [-1, 1]^n \times \mathcal{F}_i$ are $C^\infty$, and vary continuously on the transverse parameter in the leafwise $C^\infty$-topology.
Automorphisms of matchbox manifolds

A “smooth matchbox manifold” $M$ is analogous to a compact manifold, with the transverse dynamics of the foliation $F$ on the Cantor-like fibers $T_i$ representing fundamental groupoid data. They naturally appear in:

- dynamical systems, as minimal sets & attractors
- geometry, as laminations
- complex dynamics, as universal Riemann surfaces
- algebraic geometry, as models for “stacks”.

**Bing Question:** For which $M$ is the group $\text{Homeo}(M)$ transitive?

**Klein Question:** Do the Riemannian symmetries of $M$ characterize it?

**Zimmer Question:** What countable groups $\Lambda$ act effectively on $M$?

**Haefliger Question:** What are the topological invariants associated to matchbox manifolds, and do they characterize them in some fashion?
Theorem [Clark & Hurder 2009] Let $\mathcal{M}$ be an orientable homogeneous smooth matchbox manifold. Then $\mathcal{M}$ is homeomorphic to a McCord (or normal) solenoid. In particular, $\mathcal{M}$ is minimal, so every leaf is dense.

When the dimension of $\mathcal{M}$ is $n = 1$ (that is, $\mathcal{F}$ is defined by a flow) then this recovers the result of Hagopian, but the proof is much closer in spirit to the later proof of this case by [Aarts, Hagopian and Oversteegen 1991].

The case where $\mathcal{M}$ is given as a fibration over $\mathbb{T}^n$ with totally disconnected fibers was proven in [Clark, 2002].

The key to the proof in the general case is the extensive use of pseudogroups and groupoids – in place of Lie group actions.
Two applications

Here are two consequences of the Main Theorem:

**Corollary:** Let $\mathcal{M}$ be an orientable homogeneous $n$-dimensional smooth matchbox manifold, which is embedded in a closed $(n + 1)$-dimensional manifold. Then $\mathcal{M}$ is itself a manifold.

For $\mathcal{M}$ a homogeneous continuum with a non-singular flow, this was a question/conjecture of Bing, solved by [Thomas 1971]. Non-embedding for solenoids of dimension $n \geq 2$ was solved by [Clark & Fokkink, 2002]. Proofs use shape theory and Alexander-Spanier duality for cohomology.

**Corollary:** Let $\mathcal{M}$ be the tiling space associated to a tiling $\mathcal{P}$ of $\mathbb{R}^n$. If $\mathcal{M}$ is homogeneous, then the tiling is periodic.
Generalized solenoids

Let $M_\ell$ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$
p_{\ell+1} \to M_\ell \to M_{\ell-1} \to \cdots \to M_1 \to M_0
$$

The $p_\ell$ are called the bonding maps for the solenoid

$$
S = \lim_{\leftarrow} \\{ p_\ell : M_\ell \to M_{\ell-1} \} \subset \prod_{\ell=0}^{\infty} M_\ell
$$

Choose basepoints $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$. Set $G_\ell = \pi_1(M_\ell, x_\ell)$. Then we have a descending chain of groups and injective maps

$$
p_{\ell+1} \to G_\ell \to G_{\ell-1} \to \cdots \to G_1 \to G_0
$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 : M_\ell \to M_0$. 

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**Definition:** $S$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $H_\ell$ of $G_\ell$ in $H_{\ell_0} \equiv G_{\ell_0}$ is a normal subgroup.

**Theorem** [McCord 1965] A McCord solenoid $S$ is an orientable homogeneous smooth matchbox manifold.

**Remark:** $\pi_1(M_0, x_0)$ nilpotent implies that $S$ is a McCord solenoid.

**Caution:** There are constructions of inverse limits $S$ as above where the bonding maps are not normal coverings, and the McCord condition does not hold, but $S$ is homogeneous [Fokkink & Oversteegen 2002].

Our technique of proof of the main theorem for such examples presents the inverse limit space $S$ as homeomorphic to a normal tower of coverings.
Effros Theorem

Let $X$ be a separable and metrizable topological space. Let $G$ be a topological group with identity $e$.

For $U \subseteq G$ and $x \in X$, let $Ux = \{gx \mid g \in U\}$.

**Definition:** An action of $G$ on $X$ is *transitive* if $Gx = X$ for all $x \in X$.

**Definition:** An action of $G$ on $X$ is *micro-transitive* if for every $x \in X$ and every neighborhood $U$ of $e$, $Ux$ is a neighborhood of $x$.

**Theorem** [Effros 1965] Suppose that a completely metrizable group $G$ acts *transitively* on a second category space $X$, then it acts micro-transitively on $X$.

Interpretation for compact metric spaces

The metric on the group $\text{Homeo}(X)$ for $(X, d_X)$ a separable, locally compact, metric space is given by

$$d_H(f, g) := \sup \{d_X(f(x), g(x)) \mid x \in X\}$$
$$+ \sup \{d_X(f^{-1}(x), g^{-1}(x)) \mid x \in X\}$$

Corollary: Let $X$ be a homogeneous compact metric space. Then for any given $\epsilon > 0$ there is a corresponding $\delta > 0$ so that if $d_X(x, y) < \delta$, there is a homeomorphism $h : X \to X$ with $d_H(h, id_X) < \epsilon$ and $h(x) = y$.

In particular, for a homogeneous foliated space $\mathcal{M}$ this conclusion holds.

This observation was used by [Aarts, Hagopian, & Oversteegen 1991] and [Clark 2002] in their study of matchbox manifolds.
Holonomy groupoids

Let \( \varphi_i : U_i \to [-1, 1]^n \times \mathcal{T}_i \) for \( 1 \leq i \leq \nu \) be the covering of \( M \) by foliation charts. For \( U_i \cap U_j \neq \emptyset \) we obtain the holonomy transformation

\[
h_{ji} : D(h_{ji}) \subset \mathcal{T}_i \longrightarrow R(h_{ji}) \subset \mathcal{T}_j
\]

These transformations generate the holonomy pseudogroup \( \mathcal{G}_F \) of \( M \), modeled on the transverse metric space \( \mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\nu \).

Typical element of \( \mathcal{G}_F \) is a composition, for \( \mathcal{I} = (i_0, i_1, \ldots, i_k) \) where \( U_{i_\ell} \cap U_{i_{\ell-1}} \neq \emptyset \) for \( 1 \leq \ell \leq k \),

\[
h_{\mathcal{I}} = h_{i_ki_{k-1}} \circ \cdots \circ h_{i_1i_0} : D(h_{\mathcal{I}}) \subset \mathcal{T}_{i_0} \longrightarrow R(h_{\mathcal{I}}) \subset \mathcal{T}_{i_k}
\]

\( x \in \mathcal{T} \) is a point of holonomy for \( \mathcal{G}_F \) if there exists some \( h_{\mathcal{I}} \in \mathcal{G}_F \) with \( x \in D(h_{\mathcal{I}}) \) such that \( h_{\mathcal{I}}(x) = x \) and the germ of \( h_{\mathcal{I}} \) at \( x \) is non-trivial.

We say \( \mathcal{F} \) is without holonomy if there are no points of holonomy.

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Equicontinuous matchbox manifolds

**Definition:** $\mathcal{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_I \in \mathcal{G}_\mathcal{F}$ we have

$$x, y \in D(h_I) \text{ with } d_\mathcal{F}(x, y) < \delta \implies d_\mathcal{F}(h_I(x), h_I(y)) < \epsilon$$

**Theorem:** A homogeneous matchbox manifold $\mathcal{M}$ is equicontinuous without holonomy.

The proof relies on one basic observation and extensive technical analysis.

**Lemma:** Let $h: \mathcal{M} \to \mathcal{M}$ be a homeomorphism. Then $h$ maps the leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. That is, every $h \in \text{Homeo}(\mathcal{M})$ is foliation-preserving.

**Proof:** The leaves of $\mathcal{F}$ are the path components of $\mathcal{M}$.

**Theorem:** An equicontinuous matchbox manifold $\mathcal{M}$ is minimal.
We can now state the three main structure theorems.

**Theorem 1:** Let $\mathcal{M}$ be an equicontinuous matchbox manifold without holonomy. Then $\mathcal{M}$ is homeomorphic to a solenoid

$$S = \lim \left\{ p_\ell : M_\ell \rightarrow M_{\ell-1} \right\}$$

**Theorem 2:** Let $\mathcal{M}$ be a homogeneous matchbox manifold. Then the bonding maps above can be chosen so that $q_\ell : M_\ell \rightarrow M_0$ is a normal covering for all $\ell \geq 0$. That is, $S$ is McCord.

**Theorem 3:** Let $\mathcal{M}$ be a homogeneous matchbox manifold. Then there exists a clopen subset $V \subset \mathcal{T}$ such that the restricted groupoid $\mathcal{H}(\mathcal{F}, V) \equiv \mathcal{G}_\mathcal{F}|_V$ is a group, and $\mathcal{M}$ is homeomorphic to the suspension of the action of $\mathcal{H}(\mathcal{F}, V)$ on $V$. Thus, the fibers of the map $q_\infty : \mathcal{M} \rightarrow M_0$ are homeomorphic to a profinite completion of $\mathcal{H}(\mathcal{F}, V)$. 
Let $\mathcal{M}$ be an equicontinuous matchbox manifold without holonomy.

Fix basepoint $w_0 \in \text{int}(\mathcal{T}_1)$ with corresponding leaf $L_0 \subset \mathcal{M}$.

The equivalence relation on $\mathcal{T}$ induced by $\mathcal{F}$ is denoted $\Gamma$, and we have the following subsets:

- $\Gamma_W = \{(w, w') \mid w \in W, \, w' \in \mathcal{O}(w)\}$
- $\Gamma_W^W = \{(w, w') \mid w \in W, \, w' \in \mathcal{O}(w) \cap W\}$
- $\Gamma_0 = \{w' \in W \mid (w_0, w') \in \Gamma_W^W\} = \mathcal{O}(w_0) \cap W$

Note that $\Gamma_W^W$ is a groupoid, with object space $W$. The assumption that $\mathcal{F}$ is without holonomy implies $\Gamma_W^W$ is equivalent to the groupoid of germs of local holonomy maps induced from the restriction of $\mathcal{G}_F$ to $W$. 
**Proposition:** Let $\mathcal{M}$ be an equicontinuous matchbox manifold without holonomy. Given $\epsilon_* > 0$, then there exists $\delta_* > 0$ such that:

- for all $(w, w') \in \Gamma_{\mathcal{W}}$ the corresponding holonomy map $h_{w,w'}$ satisfies $D_{\mathcal{I}}(w, \delta_*) \subset D(h_{w,w'})$
- $d_{\mathcal{I}}(h_{w,w'}(z), h_{w,w'}(z')) < \epsilon_*$ for all $z, z' \in D_{\mathcal{I}}(w, \delta_*)$.

Let $\mathcal{W} \subset \mathcal{I}_1$ be a clopen subset with $w_0 \in \mathcal{W}$. Decompose $\mathcal{W}$ into clopen subsets of diameter $\epsilon_\ell > 0$,

$$\mathcal{W} = \mathcal{W}_1^\ell \cup \cdots \cup \mathcal{W}_{\beta_\ell}^\ell$$

Set $\eta_\ell = \min \left\{ d_{\mathcal{I}}(W_i^\ell, W_j^\ell) \mid 1 \leq i \neq j \leq \beta_\ell \right\} > 0$ and let $\delta_\ell > 0$ be the constant of equicontinuity as above.
The orbit coding function

• The code space $C_\ell = \{1, \ldots, \beta_\ell\}$

• For $w \in W$, the $C_w^\ell$-code of $u \in W$ is the function $C_{w,u}^\ell : \Gamma_w \to C_\ell$ defined as: for $w' \in \Gamma_w$ set $C_{w,u}^\ell(w') = i$ if $h_{w,w'}(u) \in W_i^\ell$.

• Define $V_\ell = \left\{ u \in W_1^\ell \mid C_{w_0,u}^\ell(w') = C_{w_0,0}^\ell(w') \text{ for all } w' \in \Gamma_0 \right\}$

**Lemma:** If $u, v \in W$ with $d_\Sigma(u, v) < \delta_\ell$ then $C_{w,u}^\ell(w') = C_{w,v}^\ell(w')$ for all $w' \in \Gamma_w$. Hence, the function $C_w^\ell(u) = C_{w,u}^\ell$ is locally constant in $u$.

Thus, $V_\ell$ is open, and the translates of this set define a $\Gamma_0$-invariant clopen decomposition of $W$. 
The Thomas tube $\tilde{N}_\ell$ for $M$ is the “saturation” of $V^\ell$ by $\mathcal{F}$.

The saturation is necessarily all of $M$. But the tube structure comes with a vertical fibration, which allows for collapsing the tube in foliation charts. This is the basis of the main technical result:

**Theorem:** For $\text{diam}(V^\ell)$ sufficiently small, there is a quotient map $\Pi_\ell: \tilde{N}_\ell \rightarrow M_\ell$ whose fibers are the transversal sections isotopic to $V^\ell$, and whose base if a compact manifold. This yields compatible maps $\Pi_\ell: M \rightarrow M_\ell$ which induce the solenoid structure on $M$.

Furthermore, if $M$ is homogeneous, then $\text{Homeo}(M)$ acts transitively on the fibers of the tower induced by the maps $\Pi_\ell: M \rightarrow M_\ell$, hence the tower is normal.
**Conjecture:** Let $M$ be an equicontinuous matchbox manifold, and $V \subset \mathcal{T}$ a clopen set. Then $M$ is characterized up to homeomorphism by the restricted groupoid $\mathcal{H}(\mathcal{F}, V) \equiv G_\mathcal{F}|V$ and any partial quotient $M_\ell$.

That is, for matchbox manifolds, Kakutani equivalence implies homeomorphism (modulo some obvious additional conditions.)

This is known for flows [Dye 1957, Fokkink 1991].
Happy Birthday, Jean!

Jean Renault - Boulder 1999