Classifying Foliations

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Half a dozen reasons to study foliations . . .

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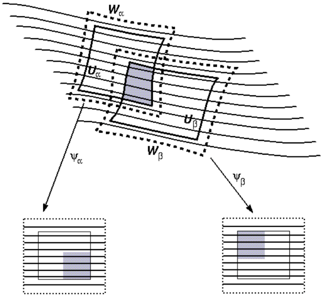
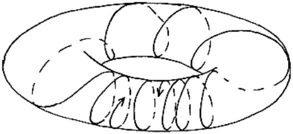
1. Generalized dynamical systems (Reeb, Godbillon, Sacksteder, Anosov, Smale, Hirsch, Shub, . . .)
2. Representation theory: cocycles, co-orbit spaces, W^* & C^*-algebras (Murray – von Neumann, Mackey, Kirillov, Kasparov, Renault, . . .)
3. Topology of classifying spaces (Bott, Haefliger, Gelfand-Fuks, Mather, Thurston, Tsuboi, . . .)
4. Geometry: laminations, 3-manifolds, isoparametric structures (Lawson, Winkelnkemper, Thurston, Gabai, Palais & Terng, . . .)
5. Physics & Non-Commutative Geometry, quasicrystals (Bellisard, Connes, . . .)
6. Descriptive Set Theory & Complexity (Kechris, Foreman, Hjorth, Louveau, Simon, . . .)

A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds of codimension q: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



A leaf of $\mathcal{F}$ is a connected component of the manifold M in the "fine" topology induced by charts.

Examples in 2-dimensions



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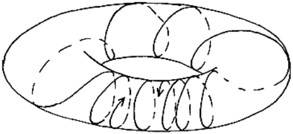
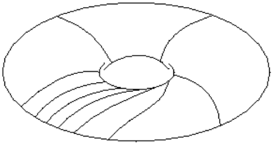
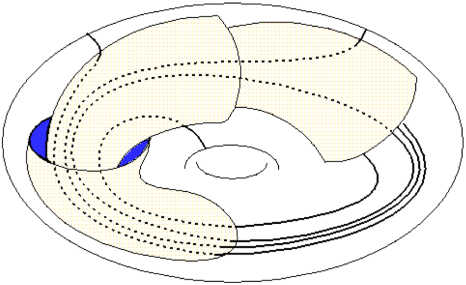




Foliations by surfaces



Reeb Foliation of the solid torus

Classifying foliations

Problem: "Classify" the foliations on a given manifold M.

Multiple classification schemes have been developed since 1970:

1. "homotopy properties" and classifying spaces;
2. "dynamical properties" and invariants;
3. "classify" von Neumann and C^*-algebras of foliations;
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Question: How are these classifications schemes related to one another?

See "Classifying foliations", to appear in Proceedings of Rio de Janeiro Conference in 2007, *Foliations, Topology and Geometry*, Contemp. Math., American Math. Soc., 2009. Or, download from website.

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- **Theorem:** (Haefliger) Each C^r-foliation $\mathcal{F}$ on M of codimension q determines a well-defined map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$ whose homotopy class in uniquely defined by $\mathcal{F}$.

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- **Theorem:** (Thurston) Each "natural" map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r \times BO_p$ corresponds to a C^r-foliation $\mathcal{F}$ on M, whose concordance class is determined by $h_{\mathcal{F}}$.

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Classification of $\mathcal{F}$ on $M \leftrightarrow$ calculate homotopy sets $[M, B\Gamma_q^r]$

How does this solve anything? Typical optimism of the 1970's.

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Classification of $\mathcal{F}$ on M $\leftrightarrow$ calculate secondary classes for examples

Last new examples by Heitsch in 1978, Hurder in 1985. Need examples!

C^2 is essential !

One clue (and caution) to the study of homotopy classification:

Theorem: (Tsuboi [1989]) The classifying map of the normal bundle $\nu \colon B\Gamma_q^1 \to BO(q)$ for foliations of transverse differentiability class C^1 is a homotopy equivalence.

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When the C^1 and C^2 situations are radically different, one asks if there is some aspects of dynamical systems involved? (There are other reasons to ask this question, too.)

Foliation dynamics

- A *continuous dynamical system* on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x, t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M, and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.

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- First step – look for topological properties of the leaves.

Limit sets

are where the orbits accumulate:

Let $\varphi_t \colon M \to M$ be a flow on a compact manifold M, and $x \in M$, then

$$\omega_x(\varphi) = \bigcap_{n=1}^{\infty} \overline{\{\varphi_t(x) \mid t \geq n\}}$$

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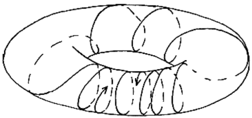
x is *recurrent* if $x \in \omega_x(\varphi)$.



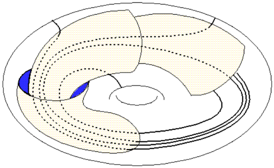
Circle is only recurrent orbit

A leaf $L \subset M$ of a foliation $\mathcal{F}$ inherits a quasi-isometry class of Riemannian metric from TM, and a metric topology.

The ω-limit set of the leaf L_x through x is

$$\omega_x(\mathcal{F}) = \bigcap_{\substack{Y \subset L_x \\ Y \text{ compact}}} \overline{L_x - Y}$$

L_x is *ω-recurrent* if $x \in \omega(L_x) \;\Rightarrow\; L_x \subset \omega(L_x)$.



Boundary torus is only recurrent leaf

Minimal sets

Dynamics on minimal sets for a foliation is first approximation to understanding its global dynamics. A closed subset $Z \subset M$ is *minimal* if

- Z is a union of leaves,
- each leaf $L \subset Z$ is dense

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For each $x \in Z$, there is a neighborhood $U_x \subset Z$

$$U_x \cong (-1, 1)^p \times K$$

where $K \subset \mathbb{R}^q$ is a closed. If K is not finite, then K is perfect.

If K has no interior points, then Z is called *exceptional*.

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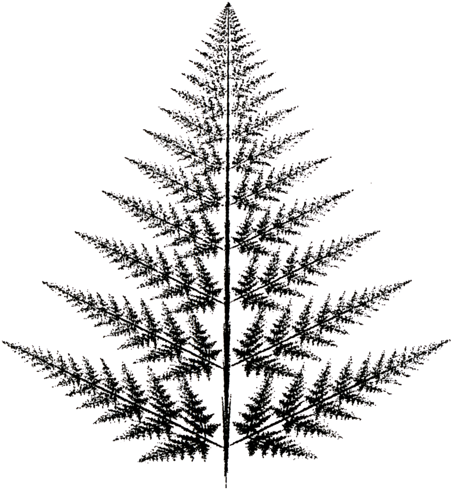
For $\mathcal{F}$ codimension $q = 1$, Z exceptional $\Leftrightarrow$ K is a Cantor set.

For $\mathcal{F}$ codimension $q \geq 2$, the possibilities for K include:

- Compact submanifolds of $\mathbb{R}^q$
- Fractals defined by *Iterated Function Systems*
- Julia sets of Rational Polynomials & Holomorphic Dynamics
- Limit sets of Schottky Groups

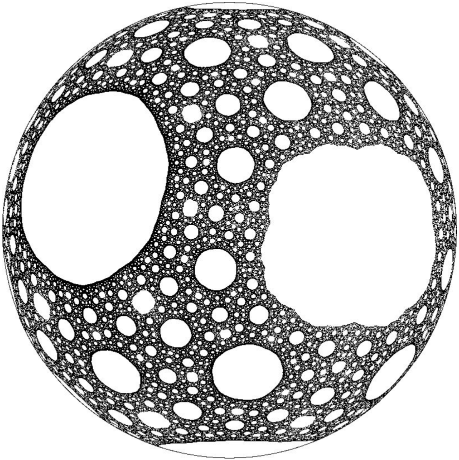
Each of these categories of "wild topology" for K is more complicated than we can hope to fully understand.

Wild Topology, I



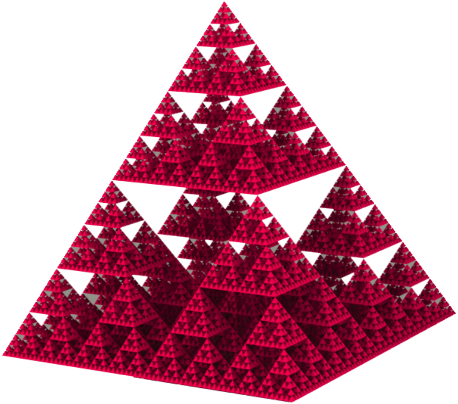
Fractal Tree

Wild Topology, II



Limit set of Schottky Group

Wild Topology, III



Sierpinski Pyramid

Shape invariants

$Z \subset M$ a minimal set of $\mathcal{F}$ always has a "neighborhood system"

$$Z \subset \cdots U_i \subset \cdots \subset U_1 \quad , \quad K = \bigcap_{i=1}^{\infty} U_i$$

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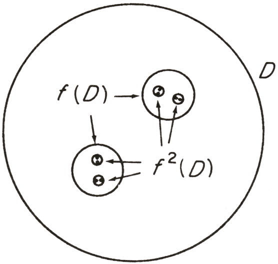
Remark: All of the minimal sets illustrated above are stable.

How to obtain an unstable minimal set?

Wild Topology, IV



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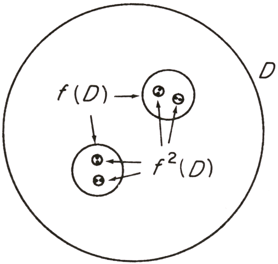

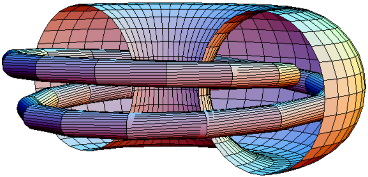

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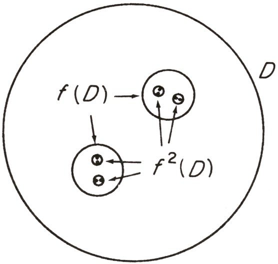
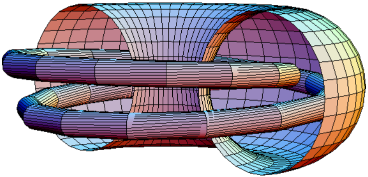


Cantor Set



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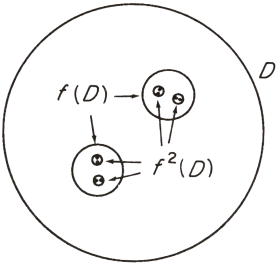

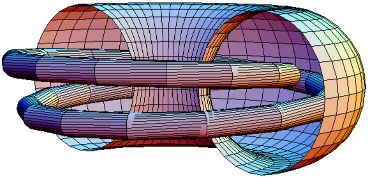

Cantor Set $\longrightarrow$ *Solenoid*

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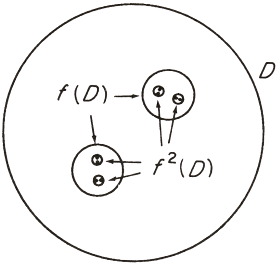

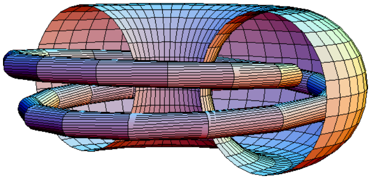

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The picture above suggests the proof that solenoids are not stable.

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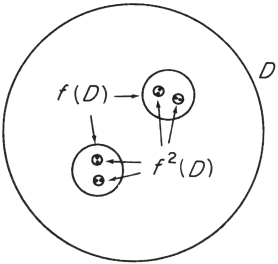

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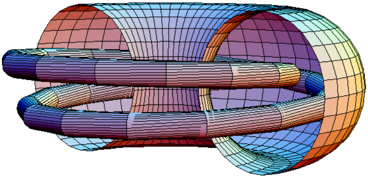

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Solenoids naturally arise as hyperbolic attractors of smooth flows.

But they are also part of the dynamics of foliations.

A construction

Theorem: [Clark & Hurder (2008)] For $p \geq 1$ and $q \geq 2n$, there exists commuting diffeomorphisms $\varphi_i \colon \mathbb{S}^q \to \mathbb{S}^q$, $1 \leq i \leq p$, so that the suspension of the induced action $\mathbb{Z}^p$ on $\mathbb{S}^q$ yields a smooth foliation $\mathcal{F}$ with solenoidal minimal set $\mathcal{S}$, such that:

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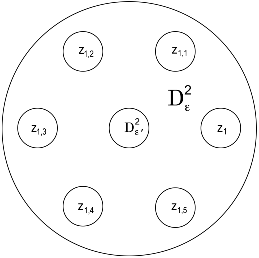
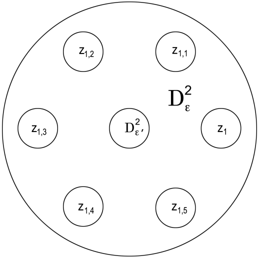
- K is an "$\{\vec{n}_i\}$-adic" completion of $\mathbb{Z}^p$: $K \cong \varprojlim \; (\Gamma_0/\Gamma_i)$.

The standard construction makes use of infinitely repeated iteration of embeddings, disks inside of disks:



First stage of inductive construction

The standard construction makes use of infinitely repeated iteration of embeddings, disks inside of disks:



First stage of inductive construction

At each stage of the iteration:

- Keep a center disk $\mathbb{D}^q_{\epsilon'}$ on which the action is a rotation about center
- View actions of $\mathbb{Z}^p$ as deformations of finite representations into $SO(q)$
- View process as sequence of inductive surgeries on suspended foliations

See "Solenoidal minimal sets for foliations", Clark & Hurder 2008

Comments

At each stage of the construction, there is a choice $\Gamma_{i+1} \subset \Gamma_i$.

For rank $p \geq 2$, the "$\vec{n}$-adic" completions of $\mathbb{Z}^p$ cannot be "classified by invariants". See "The classification problem for torsion-free abelian groups of finite rank" by S. Thomas, J. Amer. Math. Soc., 2003.

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In our construction, we force the existence of periodic disks for the action of the group $\mathbb{Z}^p$: Every open neighborhood of the minimal Cantor set K for the $\mathbb{Z}^p$-action has infinitely many essential *local actions* of finite groups whose orders tends to infinity. This encodes the algebraic data of the $\mathbb{Z}^p$-action into the dynamical data, and as homotopical data.

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An aside ...

Topological dynamics

Theorem: Let $\mathcal{F}$ be a C^r-foliation of M, $r \geq 0$. Then there is a disjoint Borel decomposition of M into $\mathcal{F}$-saturated subsets

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

• The dynamical system defined by $\mathcal{F}$ restricted to $\mathcal{E}$ is equicontinuous, and $\mathcal{E}$ is the largest such Borel saturated subset.

• The set $\mathcal{P}$ consists of the points for which the dynamical system is distal, minus the equicontinuous set.

• The set $\mathcal{H}$ is the complementary set of points where the foliation dynamical system is proximal.

In the case where $r \geq 1$, $\mathcal{H}$ can be considered as the set of points where the dynamics are *non-uniformly partially hyperbolic*.

Junk dynamics

The sets $\mathcal{E}$ and $\mathcal{P}$ are traditionally considered part of the "junk" dynamics of $\mathcal{F}$, while the hyperbolic set $\mathcal{H}$ is the "active" part of the dynamics.

Theorem: [Hurder, 1987, ..., 2007] Secondary cohomology invariants of $\mathcal{F}$ are "supported" on $\mathcal{H}$.

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But are the non-hyperbolic sets "junk"? Hardly. There is lots of cohomology data to be extracted from the "junk".

Classifying spaces

"Milnor join" construction $\Rightarrow$ classifying space BG of a Lie group G

Generalized by [Haefliger (1970), Segal (1975)] to a classifying space for a topological groupoid Γ. The space $B\Gamma \equiv \|\Gamma\|$ is the "semi-simplicial fat realization" of the groupoid Γ.

In general, the space $B\Gamma$ is as obscure as the nomenclature suggests.

Most well-known: the "universal classifying space" of codimension-q foliations, $B\Gamma_q^r$ introduced above. The objects of Γ_q^r are points of $\mathbb{R}^q$, and morphisms are germs of local C^r-diffeomorphisms of $\mathbb{R}^q$.

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The foliation $\mathcal{F}$ on M has a well-defined homotopy class $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$.

For any open set $U \subset M$, the restriction $\mathcal{F} \mid U$ defines a groupoid $\Gamma_{U|\mathcal{F}}$.

There is a natural map of its realization $B\Gamma_{U|\mathcal{F}} \to B\Gamma_q^r$.

Exotic cohomology

A neighborhood system of a minimal set $Z \subset M$ yields directed system of spaces $\{B\Gamma_{U_{i+1}|\mathcal{F}} \longrightarrow B\Gamma_{U_i|\mathcal{F}}\}_{i=1}^{\infty}$.

Definition: $\mathcal{H}^*(Z, \mathcal{F}) \equiv \lim_{\rightarrow} \{H^*(B\Gamma_{U_i|\mathcal{F}}) \to H^*(B\Gamma_{U_{i+1}|\mathcal{F}})\}$

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Example: Suppose that Z is a periodic orbit of a flow, defined by the suspension of an effective $\Gamma = \mathbb{Z}/p\mathbb{Z}$-action on a disk $\mathbb{D}^2$ fixing the origin. This is a finite group action, so is an example of "junk" dynamics. However,

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$$\mathcal{H}(Z, \mathcal{F}; \mathbb{Z}/p\mathbb{Z}) \ = \ H^*(B\Gamma; \mathbb{Z}/p\mathbb{Z}) \ \cong \ (\mathbb{Z}/p\mathbb{Z})[e_1]$$

is a polynomial ring generated by the Euler class e_1 of degree 2.

This is just the "Borel construction" for the finite group action: a fixed-point for a group action is not really a point; in homotopy theory, it is BG_x where G_x is the isotropy subgroup of the fixed-point x.

Application

Each open neighborhood U_i of the minimal set Z yields is a natural map $B\Gamma_{U_i|\mathcal{F}} \to B\Gamma_q$, hence an induced map of the limit space

$$h_Z \colon \widehat{Z} \equiv \lim_{\leftarrow} \ \{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\} \longrightarrow B\Gamma_q$$

Theorem: [Hurder (2008)] Let $\mathcal{S}$ be the solenoidal minimal set above. Then the homotopy class of the induced map $h_{\mathcal{S}} \colon \widehat{\mathcal{S}} \to B\Gamma_q$ is non-trivial:

$$h_{\mathcal{S}}^* \colon H^{4\ell-1}(B\Gamma_q;\mathbb{R}) \to \mathcal{H}^{4\ell-1}(\mathcal{S},\mathcal{F};\mathbb{R}) \text{ is non-trivial for } \ell > q/2$$

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Proof: The Cheeger-Simons classes for C^r-foliations, $r \geq 2$, derived from $H^*(BSO(q); \mathbb{R})$ are in the image of $h_{\mathcal{S}}^*$. ...

Cheeger-Simons classes and solenoids

Question: How can these higher dimensional classes Cheeger-Simons classes be non-zero for a solenoid defined by a flow on a 3-manifold? Or an $\mathbb{R}^p$-action on $\mathbb{T}^p \times \mathbb{D}^q$?

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Answer: For a solenoid $\mathcal{S}$ defined by an action of $G = \mathbb{Z}^p$ on $\mathbb{D}^q$, and an open neighborhood $\mathcal{S} \subset U$, the realization $B\Gamma_{U|\mathcal{F}}$ contains a copy of a Borel space BG_x for each fixed point $x \in U$ with finite group action germ. That is, the neighborhood of $\widehat{\mathcal{S}}$ in $B\Gamma_q^2$ contains infinitely many copies of Borel spaces for finite group actions. In the limit, we obtain $\mathbb{R}$-valued Cheeger-Simons classes supported on the limit $\widehat{\mathcal{S}}$, which is a *shape cycle*:

$\widehat{\mathcal{S}}$ is a *semi-simplical measured lamination* equipped with a foliated microbundle structure, carrying non-trivial cohomology classes of $B\Gamma_q^2$.

An illustration

So, applying the classifying space functor yields a nested sequence of "curled up leaves" clustering on the central stalk $\widehat{\mathcal{S}}$. How to picture this?

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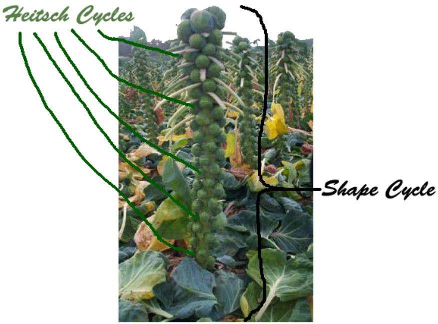

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Cheeger-Simons classes

Let $p \colon \mathbb{E} \to M$ be an oriented $\mathbb{R}^{2m}$-vector bundle over manifold M.

$\nu_{\mathbb{E}} \colon M \to B\Gamma^+_{2m} \to BSO(2m)$ is the classifying map for $E_{\mathcal{F}}$.

For example, $m = 1$ then $BSO(2) \cong \mathbb{S}^\infty/\mathbb{S}^1$.

$H^*(BSO(2m); \mathbb{R})$ is generated as algebra by:

$$
\begin{aligned}
e \in H^{2m}(BSO(2m); \mathbb{R}) \quad &- \quad \text{The Euler Class} \\
p_\ell \in H^{4\ell}(BSO(2m); \mathbb{R}) \quad &- \quad \text{The Pontrjagin Classes}
\end{aligned}
$$

Set $e(\mathbb{E}) = \nu^*_{\mathbb{E}}(e) \in H^{2m}(M; \mathbb{R})$ and $p_\ell(\mathbb{E}) = \nu^*_{\mathbb{E}}(p_\ell) \in H^{4\ell}(M; \mathbb{R})$

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Consider the Bockstein maps:

$$\cdots \to H^{*-1}(M;\mathbb{R}/\mathbb{Z}) \to H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{R}) \to \cdots$$

$C \in \ker\{H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{R})\}$, "preimage" $\widehat{C} \in H^{*-1}(M;\mathbb{R}/\mathbb{Z})$

Let $J = (j_1 \leq j_2 \leq \cdots \leq j_k)$ and set $p_J = p_{j_1} p_{j_2} \cdots p_{j_k}$.

$|J| = j_1 + \cdots j_k$ and then $\deg p_J = 4|J|$.

In the case where $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers, and $C = p_J(\mathbb{E})$ with $|J| > m$, we have:

Bott Vanishing Theorem: [1970] If $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers of $p \colon \mathbb{E} \to M$, then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{R}) \to H^*(M; \mathbb{R}) \text{ is trivial for } * > 2q = 4m.$$

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The Bott Vanishing Theorem implies there exists pre-images $\widehat{p_J(\mathbb{E})} \in H^{4|J|-1}(M; \mathbb{R}/\mathbb{Z})$. If $\mathbb{E}$ is a trivial bundle, then these "Bockstein classes" lift to the *Cheeger-Simons classes* for $\mathcal{F}$:

$$\begin{aligned} T(p_J) &\in H^{4|J|-1}(M; \mathbb{R}) \\ T(e_m) &\in H^{2m-1}(M; \mathbb{R}) \end{aligned}$$

Generalized winding numbers

For $m = 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^2$-bundle $\mathbb{E} \to M$, then have

$$T(e_1^\ell) \in H^{2\ell-1}(M; \mathbb{R}) \ , \ \ell > 2$$

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For $m > 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^{2m}$-bundle $\mathbb{E} \to M$, then for each $p \in H^*(BSO(2m); \mathbb{Z})$ there is a "generalized non-commutative winding invariant"

$$T(p) \in H^{*-1}(M; \mathbb{R}) \ , \ * > 4m$$

Heitsch Thesis (1970)

Theorem: The Bott Vanishing Theorem is false for $\mathbb{Z}$ coefficients!

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Example, continued: Let $(\mathbb{Z}/p\mathbb{Z})^m$ act on $\mathbb{D}^{2m}$ via rotations $\{\varphi_1, \ldots, \varphi_m\}$ with period p on each of the m-factors of $\mathbb{D}^2$.

Form the suspension flat bundle

$$\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi$$

Then the composition

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \to H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \to H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Let $p \to \infty$.

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