Classifying Foliations

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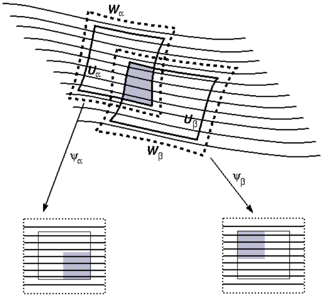
Half a dozen reasons to study foliations . . .

Origins began around 1952 with the theses of George Reeb and André Haefliger, and prior work of Charles Ehresmann. Since then many new applications and techniques have been developed:

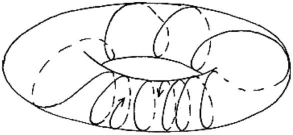
1. Generalized dynamical systems (Reeb, Godbillon, Sacksteder, Anosov, Smale, Hirsch, Shub, . . .)
2. Representation theory: cocycles, co-orbit spaces, W^* & C^*-algebras (Murray – von Neumann, Mackey, Kirillov, Kasparov, Renault, . . .)
3. Topology of classifying spaces (Bott, Haefliger, Gelfand-Fuks, Mather, Thurston, Tsuboi, . . .)
4. Geometry: laminations, 3-manifolds, isoparametric structures (Lawson, Winkelnkemper, Thurston, Gabai, Palais & Terng, . . .)
5. Physics & Non-Commutative Geometry, quasicrystals (Bellisard, Connes, . . .)
6. Descriptive Set Theory & Complexity (Kechris, Foreman, Hjorth, Louveau, Simon, . . .)

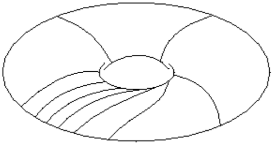
A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds of codimension q: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



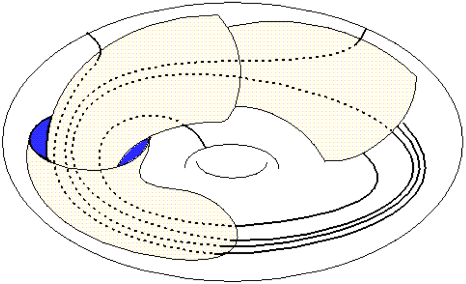
A leaf of $\mathcal{F}$ is a connected component of the manifold M in the "fine" topology induced by charts.

Examples in 2-dimensions





Foliations by surfaces



Reeb Foliation of the solid torus

Classifying foliations

Problem: "Classify" the foliations on a given manifold M.

Multiple classification schemes have been developed since 1970:

1. "homotopy properties" and classifying spaces;
2. "dynamical properties" and invariants;
3. "classify" von Neumann and C^*-algebras of foliations;
4. "complexity theory" of Borel equivalence relations.

All are long-term research topics, dating from the 1970's.

Question: How are these classifications schemes related to one another?

See "Classifying foliations", to appear in Proceedings of Rio de Janeiro Conference in 2007, *Foliations, Topology and Geometry*, Contemp. Math., American Math. Soc., 2009. Or, download from website.

Homotopy classification

- q is the codimension, p is the leaf dimension of the foliation $\mathcal{F}$.
- $B\Gamma_q^r$ denotes the "classifying space" of (smooth) codimension q-foliations with transverse differentiability C^r, introduced by André Haefliger in 1970. The homotopy fiber $F\Gamma_q^r \to B\Gamma_q^r \to BO(q)$ classifies foliations with framed normal bundles.
- **Theorem:** (Haefliger) Each C^r-foliation $\mathcal{F}$ on M of codimension q determines a well-defined map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$ whose homotopy class in uniquely defined by $\mathcal{F}$.
- **Theorem:** (Thurston) Each "natural" map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r \times BO_p$ corresponds to a C^r-foliation $\mathcal{F}$ on M, whose concordance class is determined by $h_{\mathcal{F}}$.

$$\text{Classification of } \mathcal{F} \text{ on } M \leftrightarrow \text{calculate homotopy sets } [M, B\Gamma_q^r]$$

How does this solve anything? Typical optimism of the 1970's.

Cohomology: Secondary classes

- **Theorem:** (Godbillon-Vey [1971]) For each codimension q, there is a secondary invariant $GV(\mathcal{F}) = \Delta(h_1 c_1^q) \in H^{2q+1}(M; \mathbb{R})$.
- **Theorem:** (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972]) For each codimension q, and $r \geq 2$, there is a non-trivial space of secondary invariants $H^*(W_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{tikzcd} & H^*(F\Gamma_q^r; \mathbb{R}) \arrow[d, "h_{\mathcal{F}}^*"] \\ H^*(W_q) \arrow[ur, "\tilde{\Delta}"] \arrow[r, "\Delta"] & H^*(M; \mathbb{R}) \end{tikzcd}$$

Classification of $\mathcal{F}$ on M $\leftrightarrow$ calculate secondary classes for examples

Last new examples by Heitsch in 1978, Hurder in 1985. Need examples!

C^2 is essential !

One clue (and caution) to the study of homotopy classification:

Theorem: (Tsuboi [1989]) The classifying map of the normal bundle $\nu\colon B\Gamma_q^1 \to BO(q)$ for foliations of transverse differentiability class C^1 is a homotopy equivalence.

The proof is a technical tour-de-force, using Mather-Thurston type techniques for the study of $B\Gamma_q^r$.

When the C^1 and C^2 situations are radically different, one asks if there is some aspects of dynamical systems involved? (There are other reasons to ask this question, too.)

Foliation dynamics

- A *continuous dynamical system* on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x,t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M, and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.

- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the *dynamics* of $\mathcal{F}$ asks for properties of the limiting and statistical behavior of the collection of its leaves.

- First step – look for topological properties of the leaves.

Limit sets

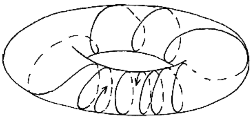
are where the orbits accumulate:

Let $\varphi_t \colon M \to M$ be a flow on a compact manifold M, and $x \in M$, then

$$\omega_x(\varphi) = \bigcap_{n=1}^{\infty} \overline{\{\varphi_t(x) \mid t \geq n\}}$$

is a compact set which is a union of flow lines for φ.

x is *recurrent* if $x \in \omega_x(\varphi)$.



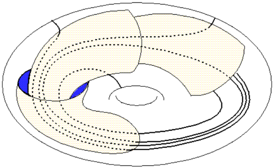
Circle is only recurrent orbit

A leaf $L \subset M$ of a foliation $\mathcal{F}$ inherits a quasi-isometry class of Riemannian metric from TM, and a metric topology.

The ω-limit set of the leaf L_x through x is

$$\omega_x(\mathcal{F}) = \bigcap_{\substack{Y \subset L_x \\ Y \text{ compact}}} \overline{L_x - Y}$$

L_x is *ω-recurrent* if $x \in \omega(L_x) \;\Rightarrow\; L_x \subset \omega(L_x)$.



Boundary torus is only recurrent leaf

Minimal sets

Dynamics on minimal sets for a foliation is first approximation to understanding its global dynamics. A closed subset $Z \subset M$ is *minimal* if

- Z is a union of leaves,
- each leaf $L \subset Z$ is dense

$$Z \text{ minimal} \quad \Rightarrow \quad \omega_x(L_x) = Z \text{ for all } x \in Z$$

First, can we describe their shape?

For each $x \in Z$, there is a neighborhood $U_x \subset Z$

$$U_x \cong (-1,1)^p \times K$$

where $K \subset \mathbb{R}^q$ is a closed. If K is not finite, then K is perfect.

If K has no interior points, then Z is called *exceptional*.

Shape of minimal sets

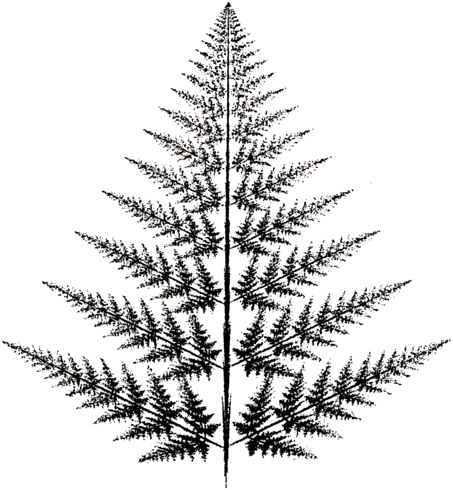
For $\mathcal{F}$ codimension $q = 1$, Z exceptional $\Leftrightarrow$ K is a Cantor set.

For $\mathcal{F}$ codimension $q \geq 2$, the possibilities for K include:

- Compact submanifolds of $\mathbb{R}^q$
- Fractals defined by *Iterated Function Systems*
- Julia sets of Rational Polynomials & Holomorphic Dynamics
- Limit sets of Schottky Groups

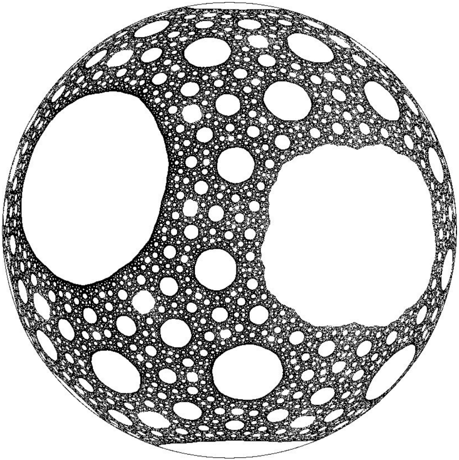
Each of these categories of "wild topology" for K is more complicated than we can hope to fully understand.

Wild Topology, I



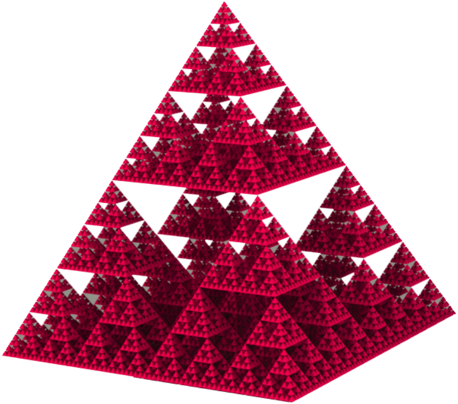
Fractal Tree

Wild Topology, II



Limit set of Schottky Group

Wild Topology, III



Sierpinski Pyramid

Shape invariants

$Z \subset M$ a minimal set of $\mathcal{F}$ always has a "neighborhood system"

$$Z \subset \cdots U_i \subset \cdots \subset U_1 \quad , \quad K = \bigcap_{i=1}^{\infty} U_i$$

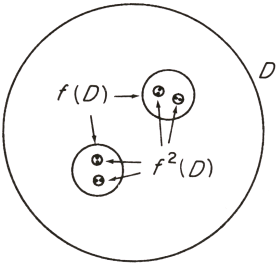
where the U_i are open. The system defines the *shape* of K.

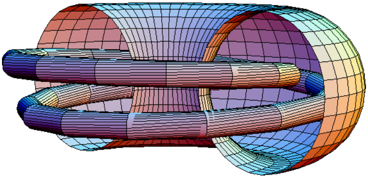
Definition: A minimal set $Z \subset M$ is *moveable* (or *stable*) if it has a neighborhood system such that for $i \gg 1$ the inclusions $U_{i+1} \subset U_i$ are homotopy equivalences.

Remark: All of the minimal sets illustrated above are stable.

How to obtain an unstable minimal set?

Wild Topology, IV





Cantor Set $\longrightarrow$ *Solenoid*

The leaves of $\mathcal{F}$ give a "twist" to the points of K. Because it is also a foliation, there is also a "twist" imparted to every open neighborhood of K.

The picture above suggests the proof that solenoids are not stable.

Solenoids naturally arise as hyperbolic attractors of smooth flows.

But they are also part of the dynamics of foliations.

A construction

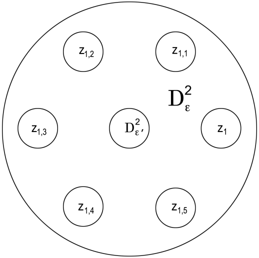
Theorem: [Clark & Hurder (2008)] For $p \geq 1$ and $q \geq 2n$, there exists commuting diffeomorphisms $\varphi_i \colon \mathbb{S}^q \to \mathbb{S}^q$, $1 \leq i \leq p$, so that the suspension of the induced action $\mathbb{Z}^p$ on $\mathbb{S}^q$ yields a smooth foliation $\mathcal{F}$ with solenoidal minimal set $\mathcal{S}$, such that:

- The leaves of $\mathcal{F}$ restricted to $\mathcal{S}$ are all isometric to $\mathbb{R}^p$
- Action of $\mathbb{Z}^p$ on Cantor set $K = \mathcal{S} \cap \mathbb{S}^q$ has a unique invariant probability measure (action is equivalent to generalized odometer)
- Every open neighborhood of K contains periodic <u>domains</u> for the action of $\mathbb{Z}^p$ on $\mathbb{S}^q$
- The isotropy groups of periodic orbits form a profinite series

$$\cdots \Gamma_i \subset \cdots \Gamma_1 \subset \Gamma_0 = \mathbb{Z}^n \quad , \quad \Gamma_i = \vec{n}_i \cdot \mathbb{Z}^p$$

- K is an "$\{\vec{n}_i\}$-adic" completion of $\mathbb{Z}^p$: $K \cong \varprojlim \, (\Gamma_0/\Gamma_i)$.

The standard construction makes use of infinitely repeated iteration of embeddings, disks inside of disks:



First stage of inductive construction

At each stage of the iteration:

- Keep a center disk $\mathbb{D}^q_{\epsilon'}$ on which the action is a rotation about center
- View actions of $\mathbb{Z}^p$ as deformations of finite representations into $SO(q)$
- View process as sequence of inductive surgeries on suspended foliations

See "Solenoidal minimal sets for foliations", Clark & Hurder 2008

Comments

At each stage of the construction, there is a choice $\Gamma_{i+1} \subset \Gamma_i$.

For rank $p \geq 2$, the "$\vec{n}$-adic" completions of $\mathbb{Z}^p$ cannot be "classified by invariants". See "The classification problem for torsion-free abelian groups of finite rank" by S. Thomas, J. Amer. Math. Soc., 2003.

Problem: Can one "classify", in the sense of descriptive set theory, the solenoidal minimal sets?

The study of "unstable" minimal sets for foliations has great complexity.

In our construction, we force the existence of periodic disks for the action of the group $\mathbb{Z}^p$: Every open neighborhood of the minimal Cantor set K for the $\mathbb{Z}^p$-action has infinitely many essential *local actions* of finite groups whose orders tends to infinity. This encodes the algebraic data of the $\mathbb{Z}^p$-action into the dynamical data, and as homotopical data.

An aside . . .

Topological dynamics

Theorem: Let $\mathcal{F}$ be a C^r-foliation of M, $r \geq 0$. Then there is a disjoint Borel decomposition of M into $\mathcal{F}$-saturated subsets

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

• The dynamical system defined by $\mathcal{F}$ restricted to $\mathcal{E}$ is equicontinuous, and $\mathcal{E}$ is the largest such Borel saturated subset.

• The set $\mathcal{P}$ consists of the points for which the dynamical system is distal, minus the equicontinuous set.

• The set $\mathcal{H}$ is the complementary set of points where the foliation dynamical system is proximal.

In the case where $r \geq 1$, $\mathcal{H}$ can be considered as the set of points where the dynamics are *non-uniformly partially hyperbolic*.

Junk dynamics

The sets $\mathcal{E}$ and $\mathcal{P}$ are traditionally considered part of the "junk" dynamics of $\mathcal{F}$, while the hyperbolic set $\mathcal{H}$ is the "active" part of the dynamics.

Theorem: [Hurder, 1987, . . . , 2007] Secondary cohomology invariants of $\mathcal{F}$ are "supported" on $\mathcal{H}$.

The study of non-uniformly partially hyperbolic dynamics of flows is a major area of current research in smooth dynamical systems.

Solenoidal minimal sets are contained in the equicontinuous set $\mathcal{E}$.

Their shape neighborhoods are often contained in $\mathcal{E} \cup \mathcal{P}$.

But are the non-hyperbolic sets "junk"? Hardly. There is lots of cohomology data to be extracted from the "junk".

Classifying spaces

"Milnor join" construction $\Rightarrow$ classifying space BG of a Lie group G

Generalized by [Haefliger (1970), Segal (1975)] to a classifying space for a topological groupoid Γ. The space $B\Gamma \equiv \|\Gamma\|$ is the "semi-simplicial fat realization" of the groupoid Γ.

In general, the space $B\Gamma$ is as obscure as the nomenclature suggests.

Most well-known: the "universal classifying space" of codimension-q foliations, $B\Gamma_q^r$ introduced above. The objects of Γ_q^r are points of $\mathbb{R}^q$, and morphisms are germs of local C^r-diffeomorphisms of $\mathbb{R}^q$.

The foliation $\mathcal{F}$ on M has a well-defined homotopy class $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$.

For any open set $U \subset M$, the restriction $\mathcal{F} \mid U$ defines a groupoid $\Gamma_{U|\mathcal{F}}$.

There is a natural map of its realization $B\Gamma_{U|\mathcal{F}} \to B\Gamma_q^r$.

Exotic cohomology

A neighborhood system of a minimal set $Z \subset M$ yields directed system of spaces $\{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\}_{i=1}^{\infty}$.

Definition: $\mathcal{H}^*(Z, \mathcal{F}) \equiv \varinjlim \ \{H^*(B\Gamma_{U_i|\mathcal{F}}) \to H^*(B\Gamma_{U_{i+1}|\mathcal{F}})\}$

Example: Suppose that Z is a periodic orbit of a flow, defined by the suspension of an effective $\Gamma = \mathbb{Z}/p\mathbb{Z}$-action on a disk $\mathbb{D}^2$ fixing the origin. This is a finite group action, so is an example of "junk" dynamics. However,

$$\mathcal{H}(Z, \mathcal{F}; \mathbb{Z}/p\mathbb{Z}) \ = \ H^*(B\Gamma; \mathbb{Z}/p\mathbb{Z}) \ \cong \ (\mathbb{Z}/p\mathbb{Z})[e_1]$$

is a polynomial ring generated by the Euler class e_1 of degree 2.

This is just the "Borel construction" for the finite group action: a fixed-point for a group action is not really a point; in homotopy theory, it is BG_x where G_x is the isotropy subgroup of the fixed-point x.

Application

Each open neighborhood U_i of the minimal set Z yields is a natural map $B\Gamma_{U_i|\mathcal{F}} \to B\Gamma_q$, hence an induced map of the limit space

$$h_Z \colon \widehat{Z} \equiv \lim_{\leftarrow} \{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\} \longrightarrow B\Gamma_q$$

Theorem: [Hurder (2008)] Let $\mathcal{S}$ be the solenoidal minimal set above. Then the homotopy class of the induced map $h_{\mathcal{S}} \colon \widehat{\mathcal{S}} \to B\Gamma_q$ is non-trivial:

$$h_{\mathcal{S}}^* \colon H^{4\ell-1}(B\Gamma_q;\mathbb{R}) \to \mathcal{H}^{4\ell-1}(\mathcal{S},\mathcal{F};\mathbb{R}) \text{ is non-trivial for } \ell > q/2$$

Proof: The Cheeger-Simons classes for C^r-foliations, $r \geq 2$, derived from $H^*(BSO(q);\mathbb{R})$ are in the image of $h_{\mathcal{S}}^*$. ...

Cheeger-Simons classes and solenoids

Question: How can these higher dimensional classes Cheeger-Simons classes be non-zero for a solenoid defined by a flow on a 3-manifold? Or an $\mathbb{R}^p$-action on $\mathbb{T}^p \times \mathbb{D}^q$?

Answer: For a solenoid $\mathcal{S}$ defined by an action of $G = \mathbb{Z}^p$ on $\mathbb{D}^q$, and an open neighborhood $\mathcal{S} \subset U$, the realization $B\Gamma_{U|\mathcal{F}}$ contains a copy of a Borel space BG_x for each fixed point $x \in U$ with finite group action germ. That is, the neighborhood of $\widehat{\mathcal{S}}$ in $B\Gamma_q^2$ contains infinitely many copies of Borel spaces for finite group actions. In the limit, we obtain $\mathbb{R}$-valued Cheeger-Simons classes supported on the limit $\widehat{\mathcal{S}}$, which is a *shape cycle*:

$\widehat{\mathcal{S}}$ is a *semi-simplical measured lamination* equipped with a foliated microbundle structure, carrying non-trivial cohomology classes of $B\Gamma_q^2$.

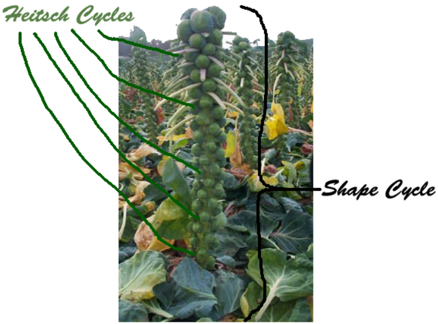
An illustration

So, applying the classifying space functor yields a nested sequence of "curled up leaves" clustering on the central stalk $\widehat{\mathcal{S}}$. How to picture this?

Typical "shape cycle"

An illustration

So, applying the classifying space functor yields a nested sequence of "curled up leaves" clustering on the central stalk $\widehat{\mathcal{S}}$. How to picture this?



Typical "shape cycle"

Cheeger-Simons classes

Let $p \colon \mathbb{E} \to M$ be an oriented $\mathbb{R}^{2m}$-vector bundle over manifold M.

$\nu_{\mathbb{E}} \colon M \to B\Gamma^+_{2m} \to BSO(2m)$ is the classifying map for $E_{\mathcal{F}}$.

For example, $m = 1$ then $BSO(2) \cong \mathbb{S}^\infty / \mathbb{S}^1$.

$H^*(BSO(2m); \mathbb{R})$ is generated as algebra by:

$$\begin{aligned} e \in H^{2m}(BSO(2m); \mathbb{R}) & \quad - \quad \text{The Euler Class} \\ p_\ell \in H^{4\ell}(BSO(2m); \mathbb{R}) & \quad - \quad \text{The Pontrjagin Classes} \end{aligned}$$

Set $e(\mathbb{E}) = \nu^*_{\mathbb{E}}(e) \in H^{2m}(M; \mathbb{R})$ and $p_\ell(\mathbb{E}) = \nu^*_{\mathbb{E}}(p_\ell) \in H^{4\ell}(M; \mathbb{R})$

Consider the Bockstein maps:

$$\cdots \to H^{*-1}(M; \mathbb{R}/\mathbb{Z}) \to H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{R}) \to \cdots$$

$C \in \ker\{H^*(M; \mathbb{Z}) \to H^*(M; \mathbb{R})\}$, “preimage” $\widehat{C} \in H^{*-1}(M; \mathbb{R}/\mathbb{Z})$

Let $J = (j_1 \leq j_2 \leq \cdots \leq j_k)$ and set $p_J = p_{j_1} p_{j_2} \cdots p_{j_k}$.

$|J| = j_1 + \cdots j_k$ and then $\deg p_J = 4|J|$.

In the case where $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers, and $C = p_J(\mathbb{E})$ with $|J| > m$, we have:

Bott Vanishing Theorem: [1970] If $\mathbb{E}$ has a foliation $\mathcal{F}$ transverse to the fibers of $p \colon \mathbb{E} \to M$, then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{R}) \to H^*(M; \mathbb{R}) \text{ is trivial for } * > 2q = 4m.$$

The Bott Vanishing Theorem implies there exists pre-images $\widehat{p_J(\mathbb{E})} \in H^{4|J|-1}(M; \mathbb{R}/\mathbb{Z})$. If $\mathbb{E}$ is a trivial bundle, then these "Bockstein classes" lift to the *Cheeger-Simons classes* for $\mathcal{F}$:

$$\begin{array}{rcl} T(p_J) & \in & H^{4|J|-1}(M; \mathbb{R}) \\ T(e_m) & \in & H^{2m-1}(M; \mathbb{R}) \end{array}$$

Generalized winding numbers

For $m = 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^2$-bundle $\mathbb{E} \to M$, then have

$$T(e_1^\ell) \in H^{2\ell-1}(M; \mathbb{R}) \ , \ \ell > 2$$

Each class $T(e_1^\ell)$ is a "generalized winding invariant" for the holonomy of the foliation $\mathcal{F}$ on the fibers of $\mathbb{E} \to M$.

For $m > 1$, $\mathcal{F}$ foliation transverse to $\mathbb{D}^{2m}$-bundle $\mathbb{E} \to M$, then for each $p \in H^*(BSO(2m); \mathbb{Z})$ there is a "generalized non-commutative winding invariant"

$$T(p) \in H^{*-1}(M; \mathbb{R}) \ , \ * > 4m$$

Heitsch Thesis (1970)

Theorem: The Bott Vanishing Theorem is false for $\mathbb{Z}$ coefficients!

Example, continued: Let $(\mathbb{Z}/p\mathbb{Z})^m$ act on $\mathbb{D}^{2m}$ via rotations $\{\varphi_1, \ldots, \varphi_m\}$ with period p on each of the m-factors of $\mathbb{D}^2$.

Form the suspension flat bundle

$$\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi$$

Then the composition

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \to H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \to H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Let $p \to \infty$.

Remarks

After 40 years, the study of the classification problem for foliations has produced wide-ranging new techniques, and new perspectives on more traditional subjects and more recent topics.

The study of foliations, and the "classification problem" first formulated by Haefliger in 1970, benefits from all of these developments.

As for "What is $B\Gamma_q^r$?" it seems we are still not even close to an answer.

Thanks . . .