Solenoidal minimal sets in foliations

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Special Session on Smooth Dynamical Systems
DePaul University, Chicago, Illinois

Laminations - back to the 60's

A p-dimensional lamination $\mathcal{L}$ is a compact foliated space $\mathbb{X}$ modeled transversally on a continua: there is a compact metric space $\mathcal{T}$ and an open covering of $\mathbb{X}$ by flow-boxes

$$\{\phi_\ell \colon (-1,1)^p \times V_\ell \to U_\ell \subset \mathbb{X} \mid 1 \leq \ell \leq k\} \ , \ V_\ell \subset \mathcal{T}$$

so that if $U_k \cap U_\ell \neq \emptyset$, then $\phi_{\ell k} = \phi_\ell \circ \phi_k^{-1}$ maps open subsets of the "horizontal" slices $(-1,1)^p \times \{x\}$ to horizontal slices, where defined.

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The leaf of $\mathcal{L}$ through $x \in \mathbb{X}$ is the connected component L_x of $\mathbb{X}$ containing x, where $\mathbb{X}$ is given the fine topology generated by the open subsets of the form $\phi_\ell(W \times \{y\})$ where $W \subset (-1,1)^p$ is open and $y \in V_\ell$.

$\mathcal{L}$ is *minimal* if every leaf L_x is dense in $\mathbb{X}$ for the metric topology.

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Basic Fact: Let $\mathbb{K}$ be a minimal set for $\mathcal{F}$ then the restriction $\mathcal{F}|\mathbb{K}$ is a minimal lamination.

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Remark: For a $\mathbb{K}$ which is the minimal set of a foliation, the model space $\mathcal{T}$ has natural local embeddings into Euclidean space $\mathbb{R}^q$ where q is the codimension of $\mathcal{F}$.

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Starter Problem: Given a class of laminations, construct foliations which have minimal sets homeomorphic to laminations in this class.

Special laminations: Solenoids

Let $L_\ell = L$ be a closed p-dimensional manifold for all $\ell \geq 0$.

Let $f\colon L \to L$ be a non-trivial covering map.

Set $f_\ell = f\colon L_\ell \to L_{\ell-1}$ for $\ell > 0$.

Definition: $\mathcal{S} = \varprojlim \{f_\ell\colon L_\ell \to L_{\ell-1}\}$ is the solenoid defined by $f\colon L \to L$.

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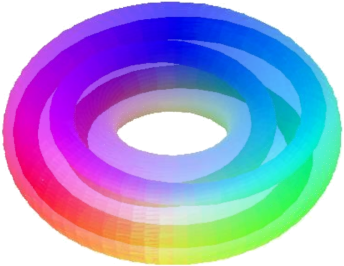
Example: $f\colon \mathbb{S}^1 \to \mathbb{S}^1$, where $f(z) = z^k$ for some integer $k > 1$.



Generalized solenoids

In the most general case, let $f_\ell \colon L_\ell \to L_{\ell-1}$ be a sequence of non-trivial covering maps of closed (branched) manifolds, for $\ell \geq 1$. Then the (generalized) solenoid defined by this data is

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Problem: Given a generalized solenoid $\mathcal{S}$ with p-dimensional leaves, when does there exists a C^r-foliation $\mathcal{F}$ of a compact manifold such that $\mathcal{S}$ is homeomorphic to a minimal set for $\mathcal{F}$?

Reeb–Thurston–Stowe Stability Theorems

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Theorem: (Reeb [1952]) Let L be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $\pi_1(L, x) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

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Theorem: (Thurston [1974]) Let L be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $H^1(L, \mathbb{R}) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

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Theorem: (Stowe [1983]) Let L be a compact leaf of a codimension q foliation $\mathcal{F}$ such that $H^1(L, \mathbb{V}) = 0$ for all flat finite-dimensional vector bundles associated to a representation of $\pi_1(L, x)$. Then there exists an open saturated neighborhood $L \subset U$ such that if $\mathcal{F}'$ is a sufficiently C^1 close to $\mathcal{F}$, then $\mathcal{F}' \mid U$ is a product foliation.

Instability of leaves

Theorem 1: (Clark & Hurder [2006]) $\mathcal{F}$ is a codimension q C^1-foliation. Let L be a compact leaf with $H^1(L, \mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. Then there exists a C^1-foliation $\mathcal{F}'$ arbitrarily C^1-close to $\mathcal{F}$ such that

- U is saturated for $\mathcal{F}'$,
- $\mathcal{F} = \mathcal{F}'$ on $M - U$,
- $\mathcal{F}' \mid U$ contains a solenoidal distal minimal set $\mathbb{K} \subset U$.

For example, if $p = 2$ and L is an oriented surface with $H^1(L; \mathbb{R}) \neq 0$, then there exists C^1-perturbations of $\mathcal{F}$ for which the leaf L has nearby solenoidal minimal sets.

Realization of solenoids

The construction technique used to prove the Theorem 1 has other applications. For example:

Fix an integer $N \geq 2$. Let Γ_ℓ be a sequence of finite groups, for $\ell \geq 1$, such that their orders satisfy a uniform bound $2 \leq |\Gamma_\ell| \leq N$.

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Theorem 2: There exists a C^0-foliation $\mathcal{F}$ of a compact manifold M with leaf dimension $p = 2$ and codimension $q = N!$ which has a compact leaf $L_0 = \Sigma_N$ (surface of genus N) and a solenoidal minimal set $\mathbb{K}$ near to L_0 so that the transverse geometry of $\mathbb{K}$ is quasi-isometric to $\mathbf{K}$.

Flat bundles

Choose a basepoint $x \in L$, and set $\Gamma = \pi_1(L, x)$.

Γ acts on the right as deck transformations of the universal cover $\widetilde{L} \to L$.

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Define a flat $\mathbb{R}^q$-bundle with holonomy ρ by

$$\mathbb{E}^q_\rho = (\widetilde{L} \times \mathbb{R}^q)/(\widetilde{y} \cdot \gamma, \vec{v}) \sim (\widetilde{y}, \gamma \cdot \vec{v}) \ \longrightarrow \ L$$

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The most familiar example is for $L = \mathbb{S}^1$ and $\Gamma = \pi_1(\mathbb{S}^1, x) = \mathbb{Z} \to \mathbf{SO(2)}$. Then $\mathbb{E}^2_\rho$ is the flat vector bundle over $\mathbb{S}^1$ with the foliation by lines of slope $\rho(1) = \exp(2\pi\sqrt{-1}\,\alpha)$.

In general, the bundle $\mathbb{E}^q_\rho \to L$ need not be a product vector bundle.

Trivializing flat bundles

Proposition: Suppose that there exists a 1-parameter family of representations $\{\rho_t : \Gamma \to \mathbf{SO(q)}\}$ such that ρ_0 is the trivial map, and $\rho_1 = \rho$, then $\{\rho_t\}$ induces a vector bundle trivialization, $\mathbb{E}^q_\rho \cong L \times \mathbb{R}^q$.

Proof: The family of representations defines a family of flat bundles $\mathbb{E}^q_{\rho_t}$ over the product space $L \times [0,1]$. This defines an isotopy between the bundles $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$, which induces a bundle isomorphism between them. The initial bundle $\mathbb{E}^q_{\rho_0}$ is a product, hence the same holds for $\mathbb{E}^q_{\rho_1}$.

In the case of the example above over $\mathbb{S}^1$ the product structure can be written down explicitly, so that we can "view" the resulting 2-torus with foliation having lines of slope α.

A key point is that the bundle isomorphism between $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$ depends smoothly on the path ρ_t.

Abelian representations

$k =$ the greatest integer such that $2k \leq q$.

$\mathbb{T}^k \subset \mathbf{SO(q)}$ a maximal embedded k-torus.

$\xi = (\xi_1, \ldots, \xi_k) \colon \Gamma \to \mathbb{R}^k$ a representation. Define

$$\begin{array}{rcl} \rho_t^\xi \colon \Gamma & \to & \mathbf{SO(q)} \\ \rho_t^\xi(\gamma) & = & [\exp(2\pi t\sqrt{-1}\xi_1(\gamma)), \ldots, \exp(2\pi t\sqrt{-1}\xi_k(\gamma))] \end{array}$$

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$$\begin{aligned} \mathbb{D}_{\epsilon}^{q} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 < \epsilon\} \subset \mathbb{R}^q \\ \mathbb{B}_{\epsilon}^{q} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 \leq \epsilon\} \subset \mathbb{R}^q \\ \mathbb{S}_{\epsilon}^{q-1} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 = \epsilon\} \subset \mathbb{R}^q \end{aligned}$$

Realizing abelian representations

Proposition: $\xi\colon \Gamma \to \mathbb{R}^k$ defines a flat bundle foliation $\mathcal{F}_\xi$ of $L \times \mathbb{S}^{q-1}$ whose leaves cover L. Moreover, if the image of ξ is contained in the rational points $\mathbb{Q}^k \subset \mathbb{R}^k$, then all leaves of $\mathcal{F}_\xi$ are compact.

Proof: ρ_t^ξ is an isotopy from ξ to the trivial representation. $\square$

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Basic Observation: Given a path $\lambda\colon [0,\epsilon] \to \mathbf{Rep}(\Gamma, \mathbf{SO(q)})$ of representations with $\lambda(\epsilon)$ the trivial representation, we obtain a foliation $\mathcal{F}_\lambda$ of $L \times \mathbb{D}^q_\epsilon$ so that

- the restriction of $\mathcal{F}_\lambda$ to the spherical fiber $L \times \mathbb{S}^{q-1}_s$ is $\mathcal{F}_{\lambda(s)}$
- the restriction of $\mathcal{F}_\lambda$ to the spherical fiber $L \times \mathbb{S}^{q-1}_\epsilon$ is the product foliation.

The basic plug

Suppose we are given a codimension-q, C^1-foliation $\mathcal{F}$, a compact leaf L with $H^1(L, \mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. We can assume that $U = L \times \mathbb{D}^q_\epsilon$.

Fix a non-trivial representation $\xi_1 \colon \Gamma \to \mathbb{Q}^k$ which exists as $H^1(L, \mathbb{R}) \neq 0$.

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Let $0 < \epsilon/2 < \epsilon_1 < \epsilon$, and set $\epsilon_1' = (\epsilon_1 + \epsilon)/2$. Choose a monotone decreasing smooth function $\mu_1 \colon [0, \epsilon] \to [0, 1]$ such that

$$\mu_1(s) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq \epsilon_1, \\ 0 & \text{if } \ \epsilon_1' \leq s \leq \epsilon \end{cases}$$

Set $\rho^{\xi_1}_{1,s} = \rho^{\mu_1(s)\xi_1} \colon \Gamma \to \mathbf{SO(q)}$. Use this family of representations to define a foliation $\mathcal{F}_1$ of $N_1 = L \times \mathbb{D}^q_\epsilon$.

Note that $\mathcal{F}_1$ is the product foliation outside of $L \times \mathbb{D}^q_{\epsilon_1'}$, and has all leaves compact inside $L \times \mathbb{B}^q_{\epsilon_1}$ and outside of $L \times \mathbb{B}^q_{\epsilon_1'}$

Iterating the plug

Let L_1 be a generic leaf of $\mathcal{F}_1$ contained in $L \times \mathbb{S}^{q-1}_{\epsilon_1/2}$.

By construction, $L_1 \to L$ is the compact covering associated to the kernel $\Gamma_1 \subset \Gamma$ of the homomorphism $\rho^{\xi_1} \colon \Gamma \to \mathbf{SO(q)}$.

Next choose $0 < \epsilon_2 < \epsilon$ sufficiently small so that $\mathcal{F}_1$ restricted to the ϵ_2-disk bundle N_2 about L_1 is a product foliation.

We now repeat the construction: choose a non-trivial map $\xi_2 \colon \Gamma_1 \to \mathbb{Q}^k$ and maps μ_2 as before.

Iterating the plug 2

Iterate for all $n \geq 2$. This yields:

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- Smooth foliations $\mathcal{F}'_n$ of $L \times \mathbb{D}^q_\epsilon$, such that all leaves of the restriction $\mathcal{F}'_n | \mathbf{K}_n$ are coverings of L that have increasingly high order, bounded below by the orders of the subgroups Γ_n.

The perturbation $\mathcal{F}'$

Proposition: If the maps ξ_n are suitably chosen (i.e., the images of the generators of Γ approach 0 in $\mathbb{Q}^k$ sufficiently rapidly) then:

1. the foliations $\mathcal{F}'_n$ converge to a C^r-foliation $\mathcal{F}'$ of $L \times \mathbb{D}^q_\epsilon$.
2. $\mathbf{K} = \cap_{n=1}^{\infty} \mathbf{K}_n$ is a saturated compact set.
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Question: If $\mathcal{F}$ is C^1, must the orders of the quotient groups Γ_n/Γ_{n+1} be unbounded?

Remark: The key property used above is that there is a representation $\rho \colon \Gamma = \pi_1(L) \to \mathbf{SO(q)}$ that is connected to the identity. This suggest the dichotomy: either $\Gamma = \pi_1(L)$ is a Kahzdan group, or there exists perturbations of $\mathcal{F}$ with solenoidal minimal sets. No idea how to do this.