The dynamics of flows in 3-dimensions

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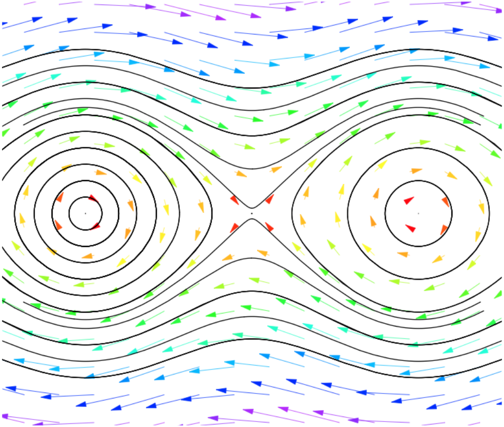
Flows in 2 dimensions

A planar vector field assigns to each point in some open domain $U \subset \mathbb{R}^2$ a vector: for $x \in U$, $x \mapsto \vec{V}(x) \in \mathbb{R}^2$.



The *flow* of a vector field $\vec{V}(x)$ on U is a path $\sigma\colon (-\epsilon, \epsilon) \to U$ such that

$$\sigma'(t) = \frac{d}{dt}\sigma(t) = \vec{V}(\sigma(t)) \quad \text{for all } -\epsilon < t < \epsilon$$

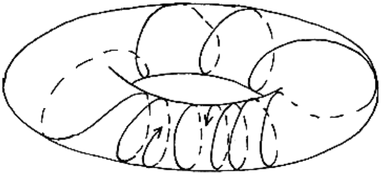
This is a *local solution* of the first-order differential equation $x'(t) = \vec{V}(x(t))$. A solution is *global* if the local solution is defined for all $-\infty < t < \infty$.

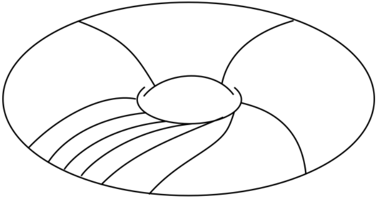
Problem: What are the global solutions of a given equation, and how do their orbits behave as time tends to ∞?

Global solutions are usually not defined, unless we impose more conditions.

For example, assume that $\vec{V}(x)$ is defined on a closed space, so the solutions cannot escape.

Here are two flows on the closed manifold $\mathbb{T}^2$





Poincaré-Bendixon Theorem, 1901: Suppose that $\vec{V}$ is a *non-vanishing* C^1-vector field on $\mathbb{T}^2$. Then either each global solution accumulates on some embedded circles in $\mathbb{T}^2$, or (in the C^2 case) each global solution accumulates on all of $\mathbb{T}^2$.

Poincaré-Hopf Theorem, 1885,1926: The only closed 2-manifold that admits a non-vanishing vector field is $\mathbb{T}^2$.

If we drop the requirement that the vector field is non-vanishing, then every closed surface admits a vector field that vanishes at most a finite number of points, where the number is the Euler characteristic of the surface. The Poincaré-Bendixon Theorem also applies in this case, but the description of the asymptotic solutions are more complicated, as they have Morse-Smale dynamics.

Flows in 3 dimensions

A 3-dimensional vector field assigns to each point in some open domain $U \subset \mathbb{R}^3$ a vector: for $x \in U$, $x \mapsto \vec{V}(x) \in \mathbb{R}^3$.

The asymptotic behavior of the solution curves in this case defy abstract description, unless severe restrictions are placed on the vector field – for example, that they are Hamiltonian.

Perhaps the most famous example is the the *Lorenz Attractor*, introduced by Edward Lorenz in 1963, as a simplified mathematical model for atmospheric convection:

Equations for the Lorenz dynamical system in $\mathbb{R}^3$

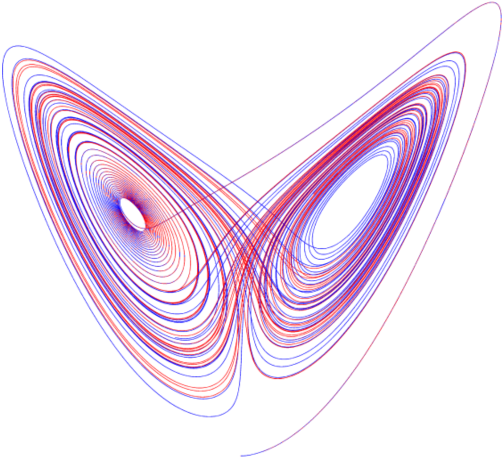
$$\begin{aligned}\frac{dx}{dt} &= \sigma \cdot (y - x)\\ \frac{dy}{dt} &= x \cdot (\rho - z) - y\\ \frac{dz}{dt} &= x \cdot y - \beta \cdot z\end{aligned}$$

where σ, ρ, β are system parameters.

In the 1970's computer time became sufficiently inexpensive, so that computer models of the solutions could be made during the evenings when mainframe computers were made available to staff.

This led to a big surprise:

Typical solution of the Lorenz differential equations in $\mathbb{R}^3$



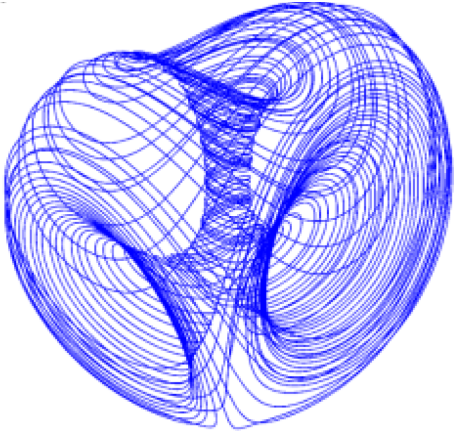
There are many variants of the Lorenz type systems in $\mathbb{R}^3$.

For example, the Dequan-Li system is given by

$$\begin{aligned}
\frac{dx}{dt} &= \alpha \cdot (y - x) + \delta \cdot x \cdot y \\
\frac{dy}{dt} &= \rho \cdot x + \zeta \cdot y - x \cdot z \\
\frac{dz}{dt} &= \beta \cdot z + x \cdot y - \epsilon \cdot x^2
\end{aligned}$$

where $\alpha, \beta, \delta, \rho, \zeta,$ are system parameters.

A solution of the Dequan-Li system



Model of the Dequan-Li Attractor by the artist “Istvan”

How to *mathematically* study such dynamical systems?

Introduce a Poincaré cross-section to the flow: a surface $\Sigma \subset \mathbb{R}^3$ transverse to the orbits of the flow, and study the behavior of the return points of the flow to the surface.

This generates a 2-dimensional dynamical system $F\colon \Sigma' \to \Sigma$ where $\Sigma' \subset \Sigma$ is an open subset whose points return to the section under the flow.

Definition: A point is *non-wandering* if its future and past orbits return infinitely often to a neighborhood of the point.

$\Omega \subset \Sigma'$ denotes the set of all non-wandering points for the system.

Main Problem: Describe the non-wandering set Ω.

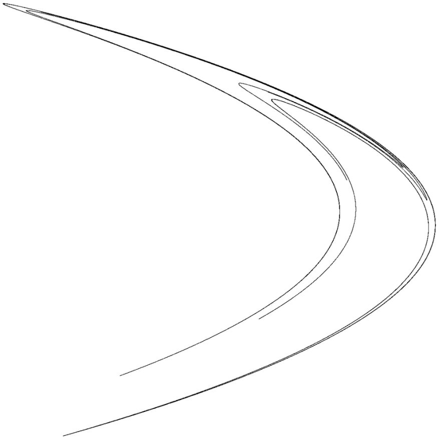
The 2-dimensional model for the Lorenz Attractor is the Hénon dynamical system:

The recursive definition of points in an orbit are given by $(x_{n+1}, y_{n+1}) = F(x_n, y_n)$

$$\begin{aligned} x_{n+1} &= 1 - a \cdot x_n^2 + b \cdot y_n \\ y_{n+1} &= x_n \end{aligned}$$

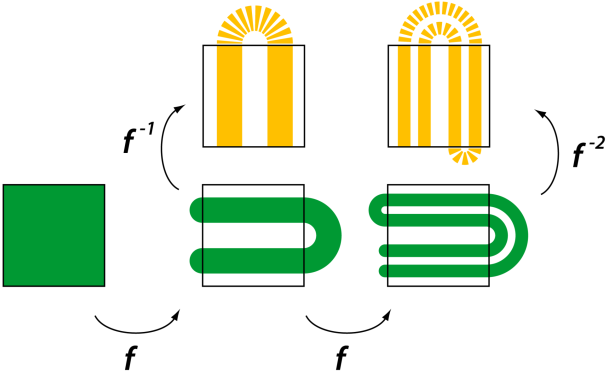
where a, b are system parameters.

Hénon Attractor is the planar model for the Lorenz Attractor



What is the non-wandering set Ω for this system?

It is a *horseshoe dynamical system*, for appropriate choices of the parameters a, b



From 1970 till present, the study of the chaotic dynamics of flows, especially in 3-dimensions, is a principle focus of research in dynamics. Thousands of papers written on this topic.

For an introduction to this approach to dynamics, see the work:

Henk Broer & Floris Takens, **Dynamical systems and chaos**, Applied Mathematical Sciences, Vol. 172. Springer, 2011.

Broer and Takens were the doctoral thesis advisors for Olga Lukina.

If you can't say something precise, conjecture something general:

Weak Palis Conjecture: M compact manifold, then every non-singular C^1-flow on M is either C^1-close to a

- Morse-Smale (gradient) flow, or
- flow which has a cross-section that exhibits horseshoe dynamics.

J. Palis, *On Open questions leading to a global perspective in dynamics*, **Nonlinearity**, 21, 2008.

This is a very Deep and Hard problem in Dynamical Systems.
It present the dream of the dynamicist mathematicians.

A proof of the WPC was posted to the arXiv by Qianying Xiao and Zuohuan Zheng last summer: arXiv:1507.07781

Aperiodic flows

In 1950 Herbert Seifert posed the:

Question: Does every non-singular vector field on the 3-sphere $\mathbb{S}^3$ have a periodic orbit?

The extension of this question to all closed 3-manifolds came to be known as the "Seifert Conjecture".

It is known that the types of maps in the Palis Conjecture all have periodic orbits, so this question is asking, what else can happen with non-singular flows on 3-manifolds?

Theorems: On any closed 3-manifold M,

- Wesley Wilson (*Annals of Mathematics* 1966)

There exists a C^∞-flow with only a finite number of periodic orbits.

- Paul Schweitzer, SJ (*Annals of Mathematics* 1974)

There exists a C^1-flow with no periodic orbits.

- Jenny Harrison (*Topology* 1988)

There exists a C^2-flow with no periodic orbits.

- Krystyna Kuperberg (*Annals of Mathematics* 1994)

There exists a C^∞-flow with no periodic orbits.

Expositions of the proof of Kuperberg's celebrated result can be found in these sources, in increasingly greater detail:

• K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.

• É. Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, 227: 283–307, 1995.

• S. Hurder & A. Rechtman, *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377 (216), 250 pages.

What we know of the dynamics of the Kuperberg flows:

Theorem (A. Katok, 1980) *Let M be a closed, orientable 3-manifold. Then an aperiodic flow ϕ_t on M has entropy zero.*

Theorem (Ghys, Matsumoto, 1995) The Kuperberg flow has a unique minimal set $\mathfrak{M} \subset M$.

Theorem (Hurder & Rechtman, 2015) *Let Φ_t be a generic Kuperberg flow on a plug $\mathbb{K}$. Then the non-wandering set Ω for the flow is equal to its unique minimal set $\mathfrak{M}$, which is a 2-dimensional lamination "with boundary".*

Moreover, the flow restricted to $\mathfrak{M}$ has non-zero "slow entropy", for exponent $\alpha = 1/2$.

So, a generic Kuperberg flow *almost* has positive entropy.

Question: Where do the Kuperberg flows sit in the scheme of the Weak Palis Conjecture?

Theorem 1: *Let Φ_t be a Kuperberg flow on a plug $\mathbb{K}$. Then there is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $-1 < \epsilon \leq 0$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ is "partially Morse-Smale" and so has entropy 0.*

Theorem 2: *Let Φ_t be a Kuperberg flow on a plug $\mathbb{K}$. Then there is a C^∞-family of flows Φ_t^ϵ on $\mathbb{K}$, for $0 \leq \epsilon < a$, with $\Phi_t^0 = \Phi_t$, such that each flow Φ_t^ϵ is chaotic with positive entropy.*

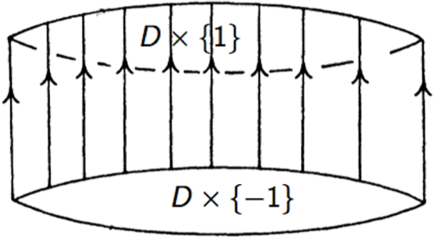
Conclusion: *The generic Kuperberg flows lie at the boundary of chaos (entropy > 0) and the boundary of tame dynamics.*

- S. Hurder & A. Rechtman, *Aperiodic flows at the boundary of chaos*, *arXiv:1603.07877*, March 2016.

Definition: A plug is a 3-manifold with boundary of the form $P = D \times [-1, 1]$ with D a compact surface with boundary. P is endowed with a non-vanishing vector field $\vec{X}$, such that:

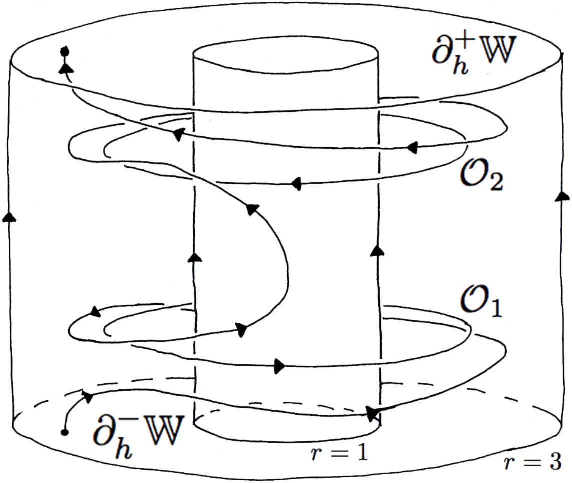
- $\vec{X}$ is vertical in a neighborhood of ∂P, that is $\vec{X} = \frac{d}{dz}$. Thus $\vec{X}$ is inward transverse along $D \times \{-1\}$ and outward transverse along $D \times \{1\}$, and parallel to the rest of ∂P.
- There is at least one point $p \in D \times \{-1\}$ whose positive orbit is trapped in P.
- If the orbit of $q \in D \times \{-1\}$ is not trapped then its orbit intersects $D \times \{1\}$ in the facing point.
- There is an embedding of P into $\mathbb{R}^3$ preserving the vertical direction.

Basic Plug - does nothing

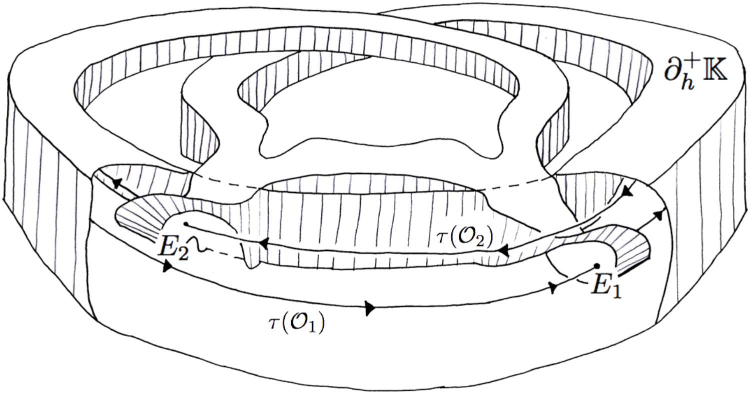


Modified Wilson Plug $\mathbb{W}$ (sort of Morse-Smale)

Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f\,\frac{f}{d\theta}$
f is asymmetric in z and $\vec{W}_1 = g\,\frac{f}{dz}$ is vertical.

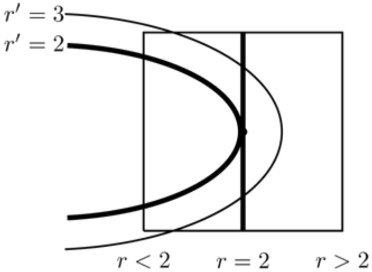


Grow horns and embed them to obtain Kuperberg Plug $\mathbb{K}$, matching the flow lines on the boundaries.



Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself.

The insertion map as it appears in the face E_1

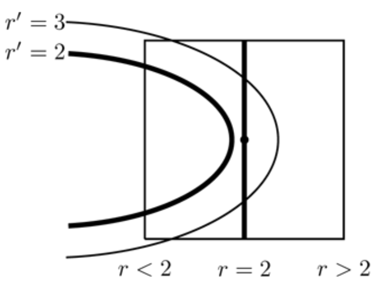


Radius Inequality:

For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^{\epsilon}(r', \theta', -2) \in \mathcal{L}_i$, then $r < r'$ unless $x' = (2, \theta_i, -2)$ and then $r = 2$.

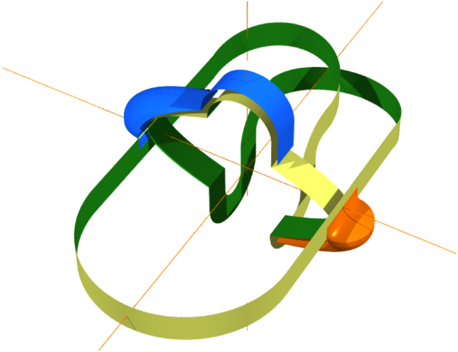
Parametrized Radius Inequality: For all $x' = (r', \theta', -2) \in L_i$, let $x = (r, \theta, z) = \sigma_i^\epsilon(r', \theta', -2) \in \mathcal{L}_i$, then $r < r' + \epsilon$ unless $x' = (2, \theta_i, -2)$ and then $r = 2 + \epsilon$.

The modified radius inequality for the cases $\epsilon < 0$ and $\epsilon > 0$:



How to study the dynamics of Kuperberg flows Φ_t^{ϵ}?

Method 1 – Have Jos Ley make computer graphic models for the flow $\Phi_t(\mathcal{R})$ of the core cylinder

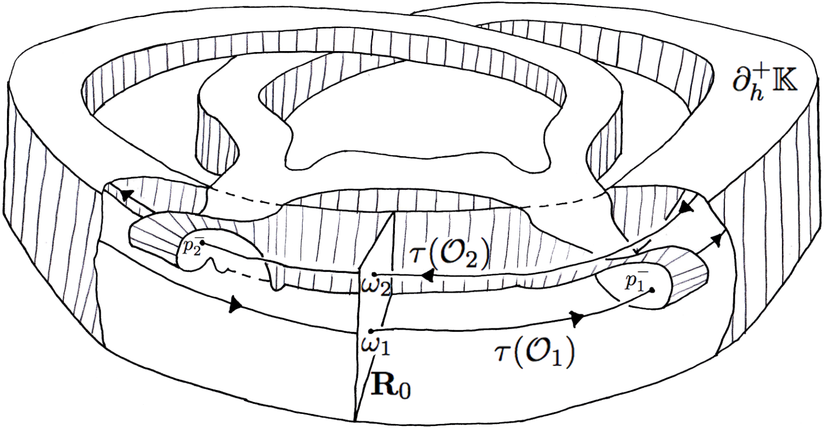


Method 2 – Introduce a cross-section $\mathbf{R}_0$ to the flow, then compare the dynamics of the flow Φ_t^0 with that of the Poincaré return map to the flow. But there are many subtleties to this method, as there is no actual cross-section to the flow which is transversal.

Return map of a flow Φ_t^ϵ induces a smooth pseudogroup $\mathcal{G}_{\Phi^\epsilon}$ on $\mathbf{R}_0$

Critical difficulty: *There is not always a direct relation between the continuous dynamics of the flow* Φ_t^ϵ *and the discrete dynamics of the action of the pseudogroup* $\mathcal{G}_{\Phi^\epsilon}$*.*

The section $\mathbf{R}_0 \subset \mathbb{K}$ used to define pseudogroup $\mathcal{G}_{\Phi^\epsilon}$.



The flow of Φ_t^ϵ is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$, so the action of the pseudogroup has singularities along this line.

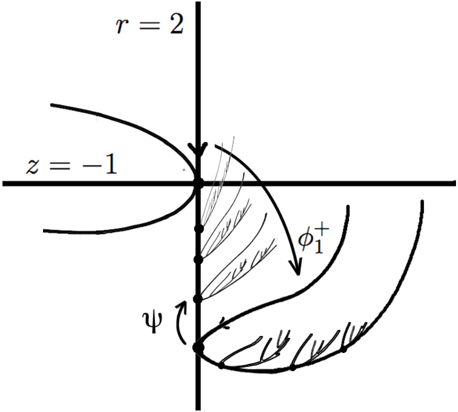
We consider two maps with domain in $\mathbf{R}_0$

- ψ which is the return map of the *Wilson flow* Ψ_t
- ϕ_1^ϵ which is the return map of the *Kuperberg flow* Φ_t^ϵ for orbits that go through the entry region E_1

Form the pseudogroup they generate $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$.

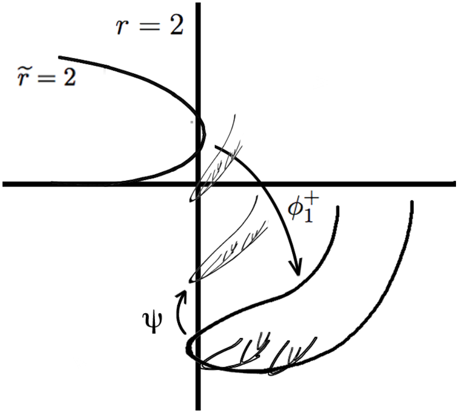
Proposition: The restriction of $\widehat{\mathcal{G}}_\epsilon$ to the region $\{r > 2\} \cap \mathbf{R}_0$ is a sub-pseudogroup of $\mathcal{G}_{\Phi^\epsilon}$

Action of $\widehat{\mathcal{G}}_0 = \langle \psi, \phi_1^\epsilon \rangle$ on the line $r = 2$ for $\epsilon = 0$.



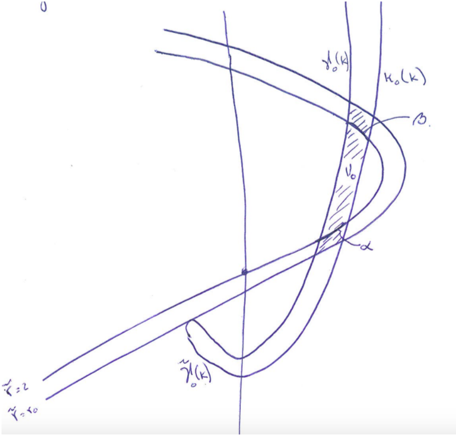
This looks like a ping-pong game, except that the play action is too slow to generate entropy.

Action of $\widehat{\mathcal{G}}_\epsilon = \langle \psi, \phi_1^\epsilon \rangle$ on the line $r = 2$ for $\epsilon > 0$.

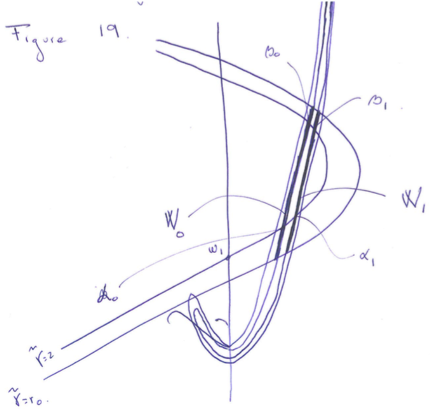


The dynamics of this action is actually too complicated to draw precisely, or calculate with.

Instead, we define a compact region $U_0 \subset \mathbf{R}_0$ which is mapped to itself by the map $\varphi = \psi^k \circ \phi_1^\epsilon$ for k sufficiently large.



The images of the powers φ^ℓ of the map the map φ form a δ-separated set for the action of the pseudogroup $\widehat{\mathcal{G}}_\epsilon$.



We then show that for $\epsilon > 0$ well-chosen with respect to the choice of k above, the restriction of the map φ to U_0 is defined by the return map of Φ_t^e and hence Φ_t^e has positive entropy.

- For $\epsilon > 0$, the dynamics of the map Φ_t^ϵ is chaotic, but making calculations of entropy for example, is only possible for well-chosen embeddings. We have no intuition, for example, of how to describe the nonwandering set for the flows Φ_t^ϵ. That is, the behavior of the chaotic orbits. We just know that some of them follow the folding map trajectories of the horseshoe dynamics.

Conclusion: The study of the dynamics for flows in 3-space is difficult, and mostly incomplete – but fascinating!

Thank you for your attention!