Classifying Foliations

after Bott, Haefliger & Thurston

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder

Séminaire général
Institut de Mathématiques de Bourgogne
November 7, 2012

Foliations...

Origins of modern theory began around with theses of Reeb [1952] and Haefliger [1956], and work of Ehresmann in 1940's and 50's. Now:

1. Generalized dynamical systems
   (Reeb, Godbillon, Sacksteder, Anosov, Smale, Hector, Ghys . . .)
2. Representation theory: cocycles, co-orbit spaces, W^* & C^*-algebras
   (Murray - von Neumann, Mackey, Kirillov, Kasparov, Renault, . . .)
3. Topology of classifying spaces
   (Bott, Haefliger, Gelfand-Fuks, Mather, Thurston, Tsuboi, . . .)
4. Physics & Non-Commutative Geometry, quasicrystals
   (Bellisard, Connes, Gambaudo, Kellendonk, Barge, . . .)
5. Descriptive Set Theory & Complexity
   (Kechris, Foreman, Hjorth, Louveau, Simon, . . .)

Classifying foliations

Problem: "Classify" the foliations on a given manifold M.

Multiple classification schemes have been developed:

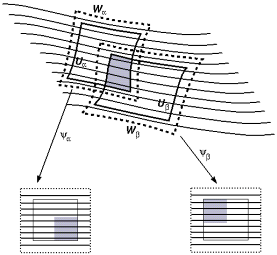
1. "homotopy properties" and classifying spaces;
2. "dynamical properties" and invariants;
3. "classify" von Neumann and C^*-algebras of foliations;
4. "complexity theory" of Borel equivalence relations.

All are long-term research topics.

Question: How are these classifications schemes related to one another?

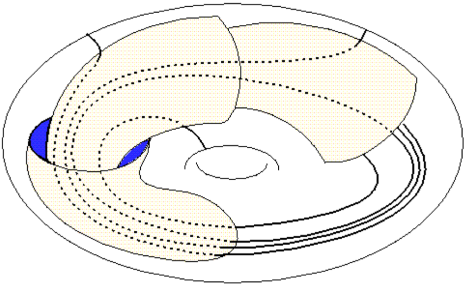
See "Classifying foliations", in *Foliations, Topology and Geometry*, Contemp. Math. vol 498, American Math. Soc., 2009.

A foliation $\mathcal{F}$ of dimension p on a manifold M^m is a decomposition into "uniform layers" – the leaves – which are immersed submanifolds of codimension q: there is an open covering of M by coordinate charts so that the leaves are mapped into linear planes of dimension p, and the transition function preserves these planes.



A leaf of $\mathcal{F}$ is a connected component of the manifold M in the "fine" topology induced by charts.

Foliations by surfaces



Reeb Foliation of the solid torus

How do you go from this seminal example, to an understanding of foliations and their geometry? Even more, how to classify?

Construct examples – surgery

Surgery on knots + Reeb foliation yields:

Theorem: [Lickorish 1965, Novikov & Zieschang 1965, Wood 1969]
Every compact 3-manifold M admits a codimension-one foliation.

Spinnable structures (existence of fibered knots) + Reeb foliation yields:

Theorem: [Durfee & H.B. Lawson 1973, Tamura 1973]
Every (m-1)-connected smooth closed (2m+1)-manifold, $m \geq 3$, admits a codimension-one foliation.

Other surgery methods for codimension-one on 3-manifolds [Gabai 1986].

Construct examples – Lie group actions

- Non-singular smooth flow $\phi_t \colon M \to M$ defines a foliation.

Same as a *locally-free* smooth action $\phi \colon \mathbb{R} \times M \to M$.

- *Locally-free* smooth action $\phi \colon \mathbb{R}^p \times M \to M$ defines foliation on M.

- G connected Lie group, *locally-free* smooth action $\phi \colon G \times M \to M$ defines foliation on M.

Example: $\Gamma \subset \mathbb{SL}(2, \mathbb{R})$ a cocompact lattice, $\mathbb{P} \subset \mathbb{SL}(2, \mathbb{R})$ the parabolic subgroup consisting of upper triangular matrices. Then $M = \mathbb{SL}(2, \mathbb{R})/\Gamma$ and $\mathbb{P}$ acts on right, to define the *Roussarie foliation* $\mathcal{F}$ of M.

Construct examples – suspensions

Given a connected, closed manifold B with $b_0 \in B$, then $\Gamma = \pi_1(B, b_0)$ is a finitely-presented group, acts freely on the universal covering $\widetilde{B} \to B$.

Given closed manifold N^q, assume there is C^r-action $\phi\colon \Gamma \times N \to N$.

$$M = \left\{ \widetilde{B} \times N \right\} / (b \cdot \gamma, x) \sim (b, \phi(\gamma) \cdot x)$$

Leaves of $\mathcal{F}_\phi$ defined by images of $\widetilde{B} \times \{x\}$ in M.

Example: Γ finitely-presented, $\phi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ a C^r-action, suspension yields a codimension-one foliation $\mathcal{F}_\phi$

These are the traditional examples...

Construct examples – inverse limits

Introduce a fourth class of examples:

Foliations obtained by *inverse limit constructions*.

In some sense, these are discretized versions of the previous examples.

Matchbox manifolds

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:
• $\mathfrak{M}$ is a continuum: a compact, connected metrizable space;
• $\mathfrak{M}$ admits a covering by foliated coordinate charts
$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^p \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$;
• each $\mathfrak{T}_i$ is a clopen subset of a totally disconnected space $\mathfrak{X}$.

Then the arc-components of $\mathfrak{M}$ are locally Euclidean:

$$\mathfrak{T}_i \quad \text{are totally disconnected} \iff \mathfrak{M} \quad \text{is a matchbox manifold}$$

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, and the pseudogroup dynamics of the foliation $\mathcal{F}$ on the transverse fibers $\mathfrak{T}_i$ represents *intrinsic* fundamental groupoid.

The "matchbox manifold" concept is much more general than minimal sets for foliations: they also appear in study of tiling spaces, subshifts of finite type, graph constructions, generalized solenoids, pseudogroup actions on totally disconnected spaces, ...

Examples:

Minimal $\mathbb{Z}^p$-actions on Cantor set K.

- Adding machines (minimal equicontinuous systems)
- Toeplitz subshifts over $\mathbb{Z}^p$
- Minimal subshifts over $\mathbb{Z}^p$
- Tiling spaces for aperiodic, repetitive tilings of $\mathbb{R}^p$ of finite local complexity.

First, a detour through the title of the talk:

"Interest level" of examples $\Leftrightarrow$ their properties + "classification theory"

Classification in the 1970's

A "new era" for foliation theory:

- Bott: *On a topological obstruction to integrability*, 1970 ICM
- Haefliger: *Feuilletages sur les variétés ouvertes,* Topology 1970.
- Bott & Haefliger: *On characteristic classes of Γ-foliations*, Bulletin AMS, 1972
- Thurston: *Theory of foliations of codimension greater than one*, Comment. Math. Helv. 1974.
- Thurston: *Existence of codimension-one foliations*, Annals of Math. 1976.
- Lawson: *Quantitative theory of foliations*, CBMS Lectures 1977.

Classification – first steps (Reeb & Haefliger)

A section $\mathcal{T} \subset M$ for $\mathcal{F}$ is an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ on $\mathcal{T}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$:

$g \in \mathcal{G}_{\mathcal{F}} \implies g \colon U \to V$ C^r-diffeomorphism, for $U, V \subset \mathcal{T}$, $r \geq 0$
$g \in \mathcal{G}_{\mathcal{F}} \implies g^{-1} \in \mathcal{G}_{\mathcal{F}}$
$g \in \mathcal{G}_{\mathcal{F}}, \text{ open } W \subset U \implies g|W \in \mathcal{G}_{\mathcal{F}}$
and elements satisfy gluing and composition rules.

Definition: $\mathcal{G}_{\mathcal{F}}$ is *compactly generated* if there is

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all orbits of $\mathcal{G}_{\mathcal{F}}$;
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ which generates $\mathcal{G}_{\mathcal{F}}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Derivative cocycle

Assume that the holonomy maps in $\mathcal{G}_{\mathcal{F}}$ are uniformly C^1.

Assume $(\mathcal{G}_{\mathcal{F}}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: *Normal derivative cocycle* $D \colon \mathcal{G}_{\mathcal{F}} \times \mathcal{T} \to \mathbb{GL}(\mathbb{R}^q)$

$$D([g]_x) = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

Satisfies the *cocycle law* (chain rule of derivatives)

$$D([h]_y \circ [g]_x) = D([h]_y) \cdot D([g]_x) \ , \ y = g(x)$$

Classifying spaces

"Milnor join" construction $\Rightarrow$ classifying space BG of a Lie group G

$B\mathbb{SO}(1) \cong B\mathbb{S}^1 \cong \mathbb{CP}^\infty$, $\quad B\mathbb{SO}(q) \cong$ infinite Grassmannian manifold of oriented q-planes in $\mathbb{R}^\infty$.

$q \geq 1$, $r \geq 1$, then Γ_q^r denotes the topological *groupoid*:

- Objects are points of $\mathbb{R}^q$.
- Morphisms are germs of local C^r-diffeomorphisms of $\mathbb{R}^q$.

[Haefliger 1970, Segal 1975] generalized Milnor join construction to obtain classifying spaces for a *topological groupoid* Γ.

$B\Gamma \equiv \|\Gamma\|$ is the "semi-simplicial fat realization" of Γ.

Homotopy classification

$B\Gamma_q^r$ is the "classifying space" of codimension-q C^r-foliations.

Derivative at a point, $g \mapsto D_x g$, defines functor, hence $\nu \colon B\Gamma_q^r \to B\mathbb{O}(q)$.

Theorem: [Haefliger 1970] Each C^r-foliation $\mathcal{F}$ on M of codimension-q determines a well-defined map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$ whose homotopy class depends only on concordance class of $\mathcal{F}$.

Remark: The homotopy fiber $F\Gamma_q^r \to B\Gamma_q^r \to BO(q)$ classifies foliations with *framed* normal bundles.

Theorem: (Bott) For $r \geq 2$ and $q \geq 2$, the fiber $F\Gamma_q^r$ is not contractible.

Theorem: (Thurston) Each "natural" map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r \times BO_p$ corresponds to a C^r-foliation $\mathcal{F}$ on M, whose concordance class is determined by $h_{\mathcal{F}}$.

Problem: How to calculate the homotopy type of the fiber $F\Gamma_q^r$?

Godbillon-Vey classes

$\mathcal{F}$ a C^2-foliation and the normal bundle $Q = TM/T\mathcal{F}$ is oriented.

ω a q-form defining $T\mathcal{F}$, then $d\omega = \eta \wedge \omega$ for a 1-form η.

The form $\eta \wedge (d\eta)^q$ is then a closed $2q+1$-form on M.

Theorem: (Godbillon-Vey [1971]) The *Godbillon-Vey class* cohomology class $GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H^{2q+1}(M; \mathbb{R})$ is independent of choices.

Theorem: [Roussarie 1972] $GV(\mathcal{F}) \neq 0$ for the Roussarie foliation.

Corollary: Homotopy type of $F\Gamma_1^r$ is very non-trivial for $r \geq 2$.

Non-vanishing of $GV(\mathcal{F})$

Question: [Moussu-Pelletier 1974, Sullivan 1975] If $GV(\mathcal{F}) \neq 0$, what does this imply about the dynamical properties of $\mathcal{F}$?

Theorem: [Duminy 1982] $GV(\mathcal{F}) \neq 0$ for C^2-foliation $\mathcal{F}$ of codimension-one $\Rightarrow$ $\mathcal{F}$ has resilient (homoclinic) leaves.

Theorem: [Hurder 1986] $GV(\mathcal{F}) \neq 0$ for C^2-foliation of codimension-q $\Rightarrow$ $\mathcal{F}$ has leaves of exponential growth.

Theorem: [Hurder & Langevin 2004] $GV(\mathcal{F}) \neq 0$ for C^1-foliation of codimension-one $\Rightarrow$ $\mathcal{F}$ has non-zero foliation entropy.

Higher secondary classes

Theorem: (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972])
For each codimension q, and $r \geq 2$, there is a non-trivial space of secondary invariants $H^*(W_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{tikzcd} & H^*(F\Gamma_q^r;\mathbb{R}) \arrow[d, "h_{\mathcal{F}}^*"] \\ H^*(W_q) \arrow[ur, "\tilde{\Delta}"] \arrow[r, "\Delta"] & H^*(M;\mathbb{R}) \end{tikzcd}$$

In this notation: $GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] = \Delta(h_1 \wedge c_1^q)$.

Classification of $\mathcal{F}$ on M $\leftrightarrow$ calculate secondary classes for examples

Non-vanishing secondary classes

Almost all examples are based on variations of a simple scheme:

Let G be a semi-simple connected Lie group.

$P \subset G$ a parabolic subgroup.

$\Gamma \subset G$ a cocompact lattice.

Then right cosets of P foliate the compact manifold $M = \Gamma / G$.

Then calculate, and obtain that the secondary classes can be non-zero over ranges of classes in $H^*(W_q)$ of the form

$$\Delta(h_1 \wedge h_{\mathcal{I}} \wedge c_{\mathcal{J}}] \text{ where } \mathcal{I} = \{i_2, \ldots, i_k\}, i_2 \geq 3, i_\ell \leq 2q \text{ odd}$$

Last new examples by Heitsch [1978], Rasmussen [1980], Hurder [1985].

Vanishing secondary classes

Theorem: [Hurder & Katok 1987] $\Delta(h_1 \wedge h_{\mathcal{I}} \wedge c_{\mathcal{J}}] \neq 0$ then the algebraic hull of the derivative cocycle cannot be *amenable* (i.e., parabolic).

There seems no hope of extending the classification scheme using Lie group examples. Theme switched in 1980's from using secondary classes to study classification problems, to showing the classification scheme fails!

Foliation dynamics

The 1980's – study of foliations as dynamical systems:

- A continuous dynamical system on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x,t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$.
- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points.
- Study the minimal sets, attractors, rates of expansion, etc. for the leaves of $\mathcal{F}$.

Closed invariant sets

As an alternative to asking for the "big picture" of foliation dynamics, study the dynamics near closed invariant sets.

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{W}$

M compact, then minimal sets for foliations always exist.

Transitive sets are most important for flows – Axiom A attractors are transitive sets, while the minimal sets include the periodic orbits in the domain of attraction.

$\mathcal{Z}$ is *exceptional minimal set* if $\mathcal{Z} \cap \mathcal{T}$ is a Cantor set.

$\mathcal{Z}$ is a matchbox manifold, embedded in a foliated neighborhood.

Structure of exceptional minimal sets

Conjecture: $\mathcal{F}$ a codimension-one, C^2-foliation, $\mathfrak{M} \subset M$ is an exceptional minimal set, then $\mathfrak{M}$ has the structure of a *Markov Minimal Set*.

That is, there is a covering of $\mathfrak{X} = \mathfrak{M} \cap \mathcal{T}$ by clopen sets, such that the restricted holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}^{\mathfrak{X}}$ is generated by a free semigroup.

Theorem: [Clark–H–Lukina, 2012] Let $\mathfrak{M}$ be an exceptional minimal set of dimension p for a C^r-foliation, $r \geq 1$. Then there exists a presentation $\mathcal{P} = \{p_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq \ell_0\}$ consisting of branched p-manifolds M_ℓ and local covering C^{r-1}-maps p_ℓ, such that $\mathfrak{M}$ is homeomorphic to the inverse system,

$$\mathfrak{M} \ \cong \ \mathcal{S}_{\mathcal{P}} = \varprojlim \ \{p_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq \ell_0\}.$$

Generalizes result for the case where $\mathfrak{M}$ is the tiling space associated to a repetitive, aperiodic tiling of $\mathbb{R}^q$ with finite local complexity.

Williams solenoids

Williams (1970, 1974) introduced a class of inverse limit spaces, called generalized solenoids, which model Smale's Axiom A attractors.

Let K be a branched manifold, i.e. each $x \in K$ has a neighborhood homeomorphic to the disjoint union of a finite number of Euclidean disks modulo some identifications.

$f : K \to K$ is an expansive immersion of branched manifolds satisfying a flattening condition. Then

$$\mathfrak{M} = \varprojlim \{f : K \to K\}$$

each point x has a neighborhood homeomorphic to $[-1,1]^n \times \mathrm{Cantor\ set}$. Moreover, f extends to a hyperbolic map defined on some open neighborhood of $\mathfrak{M} \subset V \subset \mathbb{R}^m$ for m sufficiently large.

$\mathfrak{M}$ is an exceptional minimal set for the expanding foliation $\mathcal{F}_u$ of V.

Shape

$\mathcal{Z} \subset M$ a minimal set of $\mathcal{F}$ always has a "neighborhood system"

$$\mathcal{Z} \subset \cdots U_i \subset \cdots \subset U_1 \quad , \quad K = \bigcap_{i=1}^{\infty} U_i$$

where the U_i are open. The system defines the *shape* of K.

Definition: A minimal set $\mathcal{Z} \subset M$ is *moveable* (or *stable*) if it has a neighborhood system such that for $i \gg 1$ the inclusions $U_{i+1} \subset U_i$ are homotopy equivalences.

Remark: Stable is almost the same as the tower above being sequence of homotopy equivalences.

Example: The Denjoy minimal set in $\mathbb{T}^2$ is stable.

Classification, revisted

$\mathcal{Z} \subset U \subset M$ a closed $\mathcal{F}$-saturated set $\Rightarrow$ classifying map $B\Gamma_{U|\mathcal{F}} \to B\Gamma_q$

Conjecture: Let $\mathcal{Z} \subset M$ be an exceptional minimal set which is stable, and $\mathcal{Z} \subset U$ a stable neighborhood. Then the classifying map $B\Gamma_{U|\mathcal{F}} \to B\Gamma_q$ is homotopically trivial.

In contrast, we have:

Theorem: [Hurder2012] For all $q \geq 2$, there exists open foliated spaces U containing an *unstable exceptional minimal set* $\mathcal{Z}$ such that $H^k(B\Gamma_q; \mathbb{R}) \to H^k(B\Gamma_{U|\mathcal{F}}; \mathbb{R})$ in non-zero in arbitrarily high degrees.

These examples are a new type, and are motivated by constructions in dynamical systems, not in Lie theory. Use sequence of "period doubling maps" to generate these minimal sets via limiting construction.

Bott Vanishing Theorem

We give the ideas behind this types of result.

Theorem: [Bott 1970] $\mathcal{F}$ be a C^2-foliation of codimension-q on manifold M, with oriented normal bundle. Then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m);\mathbb{R}) \to H^*(M;\mathbb{R}) \text{ is trivial for } * > 2q = 4m.$$

Corollary: Let $p \colon \mathbb{E} \to M$ be an oriented smooth vector bundle with fibers of dimension q, $\mathcal{F}$ a C^2- foliation $\mathcal{F}$ transverse to the fibers, then

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m);\mathbb{R}) \to H^*(M;\mathbb{R}) \text{ is trivial for } * > 2q = 4m.$$

Heitsch Thesis

Theorem: [Heitsch 1970] Bott Vanishing Theorem is false for $\mathbb{Z}$ coefficients! That is, there is an *injective* map

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}) \to H^*(B\Gamma_{2m}^r; \mathbb{Z})$$

Let $(\mathbb{Z}/p\mathbb{Z})^m$ act on $\mathbb{D}^{2m}$ via rotations $\{\varphi_1, \ldots, \varphi_m\}$ with period p on each of the m-factors of $\mathbb{D}^2$. Form the suspension flat bundle

$$\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi \to \mathbb{S}^\infty/(\mathbb{Z}/p\mathbb{Z}) \cong B(\mathbb{Z}/p\mathbb{Z})$$

Then the composition

$$\nu_{\mathbb{E}}^* \colon H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \to H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \to H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Then let $p \to \infty$ and use Universal Coefficient Formula.

A construction

Theorem: [Clark & Hurder 2008] For $p \geq 1$ and $q \geq 2m - 1$, there exists commuting diffeomorphisms $\varphi_i \colon \mathbb{S}^q \to \mathbb{S}^q$, $1 \leq i \leq p$, so that the suspension of the induced action $\mathbb{Z}^p$ on $\mathbb{S}^q$ yields a smooth foliation $\mathcal{F}$ with solenoidal minimal set $\mathcal{Z}$, such that:

- The leaves of $\mathcal{F}$ restricted to $\mathcal{Z}$ are all isometric to $\mathbb{R}^p$
- Action of $\mathbb{Z}^p$ on Cantor set $\mathbb{K} = \mathcal{S} \cap \mathbb{S}^q$ has a unique invariant probability measure (action is equivalent to generalized odometer)
- Every open neighborhood of $\mathbb{K}$ contains periodic domains for the action of $\mathbb{Z}^p$ on $\mathbb{S}^q$
- The isotropy groups of periodic orbits form a profinite series

$$\cdots \Gamma_i \subset \cdots \Gamma_1 \subset \Gamma_0 = \mathbb{Z}^n \quad , \quad \Gamma_i = \vec{n}_i \cdot \mathbb{Z}^p$$

- K is an "$\{\vec{n}_i\}$-adic" completion of $\mathbb{Z}^p$: $K \cong \varprojlim \, (\Gamma_0/\Gamma_i)$.

Application

Take a shape neighborhood sequence of the minimal set $\mathcal{Z}$ for the construction. Each open neighborhood U_i yields is a natural map $B\Gamma_{U_i|\mathcal{F}} \to B\Gamma_q$, hence an induced map of the limit space

$$h_Z \colon \widehat{Z} \equiv \varprojlim \ \{B\Gamma_{U_{i+1}|\mathcal{F}} \to B\Gamma_{U_i|\mathcal{F}}\} \longrightarrow B\Gamma_q$$

Theorem: The induced map $h_{\mathcal{Z}} \colon \widehat{\mathcal{Z}} \to B\Gamma_q$ is non-trivial:

$$h_{\mathcal{S}}^* \colon H^{4\ell-1}(B\Gamma_q; \mathbb{R}) \to \mathcal{H}^{4\ell-1}(\mathcal{S}, \mathcal{F}; \mathbb{R}) \text{ is non-trivial for } \ell > q/2$$

Proof: The Cheeger-Simons classes for C^r-foliations, $r \geq 2$, derived from $H^*(BSO(q); \mathbb{R})$ are in the image of $h_{\mathcal{S}}^*$.

These characteristic class of the shape of $\mathcal{Z}$ cane thought of as "non-commutative winding numbers".

Remarks

After 40 years, the study of the classification problem for foliations has produced wide-ranging new techniques, and new perspectives on more traditional subjects and more recent topics.

Relation between cohomology invariants and foliation dynamics has only been explored at the edges.

Problems:

- When does a minimal matchbox manifold embed as a minimal set in a C^r-foliation with codimension $q \geq 2$, for some $r \geq 1$?
- What are the cohomology invariants of these embeddings?

Thank you for your attention!