# Mapping class groups of solenoidal manifolds

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In the beginning...

⋆ Fokkink & Oversteegen, *Homogeneous weak solenoids*, **Trans. A.M.S.**, 354, 2002.

**Definition:** A continuum $\mathfrak{M}$ is homogeneous if for any $x, y \in \mathfrak{M}$ there is $h \in \mathrm{Homeo}(\mathfrak{M})$ with $h(x) = y$.

**Theorem.** A weak solenoid $\mathfrak{M}$ is homogeneous iff it is a normal solenoid (i.e. it is transversally modeled on a profinite group).

**Problem:** Let $\mathfrak{M}$ be a weak solenoid. Are there further relations between the structure of $\mathfrak{M}$ and $\mathrm{Homeo}(\mathfrak{M})$?

$\mathrm{Homeo}^+(\mathfrak{M})$ denotes the orientation preserving homeomorphisms.

Let $\mathrm{Homeo}_0(\mathfrak{M})$ denote the closed subgroup of $\mathrm{Homeo}^+(\mathfrak{M})$ which are isotopic to the identity map. This is a *very large* group, and is contractible. A better problem is then:

**Problem:** Let $\mathfrak{M}$ be a weak solenoid. Calculate the *Mapping Class Group*, $\mathrm{Mod}(\mathfrak{M}) = \mathrm{Homeo}^+(\mathfrak{M})/\mathrm{Homeo}_0(\mathfrak{M})$, and relate it to the structure of $\mathfrak{M}$.

**Background:** Suppose that $\Sigma_g$ is a closed surface of genus $g \geq 2$.

$\mathrm{Mod}(\Sigma_g)$ is independent of the genus for $g \geq 2$, and is often called just "the Mapping Class Group". The book

⋆ Farb & Margalit, **A Primer on Mapping Class Groups**, Princeton Mathematical Series, Vol. 49, 2012,

studies $\mathrm{Mod}(\Sigma_g)$, and gives many examples and applications.

When the compact surface $\Sigma_g$ is replaced by a weak solenoid $\mathfrak{M}$, new techniques are needed. The goal is to develop methods for calculating $\mathrm{Mod}(\mathfrak{M})$ when $\mathfrak{M}$ is a weak solenoid.

One motivation for this work comes from:

$\star$ Odden, *The baseleaf preserving mapping class group of the universal hyperbolic solenoid*, **Trans. Amer. Math. Soc.**, 357:1829–1858, 2005.

Let $\widehat{X}$ be the universal hyperbolic solenoid. That is, $\widehat{X}$ is the solenoid defined by a cofinal collection $\mathcal{P}$ in the set of all finite oriented normal coverings of $\Sigma_g$.

Let $\widehat{\Gamma}$ be the full profinite completion of $\Gamma = \pi_1(\Sigma_g, x)$.

**Theorem [Odden]:** $\mathrm{Mod}(\widehat{X}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$

**Definition:** Let $\Gamma$ be a countable group. The *abstract commensurator group* $\mathrm{Comm}(\Gamma)$ is the collection of isomorphisms $\phi\colon H \to K$ between finite-index subgroups $H, K \subset \Gamma$, modulo the equivalence which identifies isomorphisms that agree on a finite-index subgroup $L \subset H \cap K$.

**Remark:** Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Given a commensurator $\phi\colon H \to K$ in $\mathrm{Comm}(\Gamma)$, observe that $\phi\colon H \cap \Gamma' \to K \cap \Gamma'$ is a commensurator in $\mathrm{Comm}(\Gamma')$. The converse clearly holds, so there is a natural isomorphism $\mathrm{Comm}(\Gamma') \simeq \mathrm{Comm}(\Gamma)$.

The group $\mathrm{Comm}(\Gamma)$ can be intuitively viewed as the group of "germs" of isomorphisms between finite-index subgroups of G.

The work of Odden was extended by

⋆ Belk & Forrest, *Compact aspherical solenoids*, https://arxiv.org/abs/1009.5716v4.

⋆ Bering & Studenmund, *Topological models of abstract commensurators*, **Groups Geom. Dyn.**, 18:1403–1425, 2024.

Let $M$ be a closed aspherical manifold; that is, the universal covering of $M$ is contractible. Let $\widehat{M}$ be the universal solenoid, defined as the inverse limit of the system of all finite oriented normal coverings of $M$. Let $\mathcal{E}(\widehat{M})$ be the group of homotopy self-equivalences of $\widehat{M}$. Let $\widehat{\Gamma}$ be the full profinite completion of $\Gamma$.

**Theorem:** There is an isomorphism $\mathcal{E}(\widehat{M}) \to \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$.

**Problem:** Extend these results to a larger class of solenoidal manifolds, beyond the universal solenoids, and calculate $\mathrm{Mod}(\mathfrak{M})$.

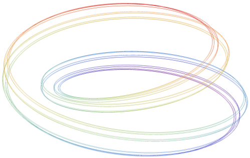
What is a solenoidal manifold?

Here is the most basic example:

Choose a sequence of integers $\vec{m} = (m_1, m_2, \ldots)$ with $m_\ell > 1$.
Form the tower of coverings

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \cdots$$

$$\mathfrak{M}_{\vec{m}} = \varprojlim \, \{m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$



Solenoidal manifolds: choose a tower of finite index coverings:

$$\mathcal{P} \equiv \qquad \{M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} M_3 \xleftarrow{p_4} \cdots\}$$

⋆ For $\ell \geq 0$, $M_\ell$ is closed, connected, $n$-dimensional manifold,

⋆ $p_{\ell+1} \colon M_{\ell+1} \to M_\ell$ is a *proper* covering map.

$$\mathfrak{M}_{\mathcal{P}} = \varprojlim \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell$$

$\widehat{q}_0 \colon \mathfrak{M}_{\mathcal{P}} \to M_0$ is fibration defined as concatenation of the maps $\{p_\ell \mid \ell \geq 0\}$, with fiber $\mathfrak{X} = \widehat{q}_0^{-1}(x_0)$ a Cantor space. $\mathfrak{M}_{\mathcal{P}}$ is:

- a *weak solenoid* (McCord) or
- a *solenoidal manifold* (Sullivan)
- *normal* if every composition $M_\ell \to M_0$ is a normal covering, which implies that $\mathfrak{X}$ is a profinite group.

**First Basic Fact:** There is a canonical homeomorphism between $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}_{\mathcal{P}_k} = \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \ , \ \ell \geq k\}$.

**Second Basic Fact:**

⋆ Rogers & Tollefson, *Homeomorphisms homotopic to induced homeomorphisms of weak solenoidal spaces*, **Colloq. Math.** 25:81–87, 1971/72.

**Theorem:** Let $\mathcal{P}$ be a tower of coverings, and $M_0$ a *strongly Borel* manifold. Then a self-homeomorphism $h \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ is isotopic to a self-homeomorphism induced by a *shuffle map*, where $i_\ell \geq \ell$ is an increasing sequence, and each $q_{i_\ell}$ is a covering map:

$$
\begin{array}{ccccccccc}
M_0 & \longleftarrow & M_{i_0} & \longleftarrow & M_{i_1} & \longleftarrow & M_{i_2} & \longleftarrow & \cdots \\
 & \swarrow_{q_{i_0}} & & \swarrow_{q_{i_1}} & & \swarrow_{q_{i_2}} & & & \\
M_0 & \longleftarrow & M_1 & \longleftarrow & M_2 & \longleftarrow & M_3 & \longleftarrow & \cdots
\end{array}
$$

Pointed maps between towers of coverings are determined by maps between fundamental groups. The problem is to understand maps between fundamental groups in a chain of subgroups.

Given a presentation $\mathcal{P}$ there are fibration maps $\widehat{p}_\ell \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$ and covering maps $q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \to M_0$.

Choose a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$, set

- $x_\ell = \widehat{p}_\ell(\widehat{x}) \in M_\ell$
- $\Gamma_\ell = (q_\ell)_\#\{\pi_1(M_\ell, x_\ell)\} \subset \Gamma = \pi_1(M_0, x_0)$
- $\mathcal{G}_{\mathcal{P}} = \{\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$

**Question:** How to construct shuffle maps for group chains?

If the shuffle maps are all inclusions, then the First Basic Fact implies the induced map of $\mathfrak{M}_{\mathcal{P}}$ is the identity. So one needs the shuffle maps to act non-trivially on fundamental groups.

The fiber $q_\ell^{-1}(x_0) \subset M_\ell$ is identified with $X_\ell = \Gamma/\Gamma_\ell$ as a $\Gamma$-space.

- $\Gamma$ acts transitively on finite set $X_\ell = \Gamma/\Gamma_\ell$
- $C_\ell \subset \Gamma_\ell$ is kernel of action map $\Phi_\ell \colon \Gamma \to \mathrm{Aut}(X_\ell)$
- $C_\ell$ is normal in $\Gamma$
- $Q_\ell = \Gamma/C_\ell$ is finite group, acting transitively on $X_\ell$.

$$X_{\mathcal{P}} \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \ .$$

$$\widehat{\Gamma}_{\mathcal{P}} \equiv \varprojlim \, \{p_{\ell+1} \colon Q_{\ell+1} \to Q_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} Q_\ell \ .$$

$\Phi \colon \Gamma \times X_{\mathcal{P}} \to X_{\mathcal{P}}$ isomorphic to monodromy of $\widehat{q}_0 \colon \mathfrak{M}_{\mathcal{P}} \to M_0$.

Action is a (generalized) *odometer*. That is, it is a minimal, equicontinuous action on a Cantor space.

Clopen set $U \subset X_{\mathcal{P}}$ is *adapted* if $U \neq \emptyset$, and for all $\gamma \in \Gamma$, $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma \cdot U = U$

$\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$ is a subgroup of finite index in $\Gamma$.

**Definition.** Odometers $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ and $\Phi' \colon \Gamma' \times \mathfrak{X}' \to \mathfrak{X}'$ are *return equivalent* if there exists adapted sets $U \subset \mathfrak{X}$ and $U' \subset \mathfrak{X}'$ and homeomorphism $h_U \colon U \to U'$ that conjugates the subgroups

$$\mathcal{H}_U = \mathrm{Image}\{\Phi_U \colon \Gamma_U \to \mathrm{Homeo(U)}\}$$
$$\mathcal{H}'_{U'} = \mathrm{Image}\{\Phi'_{U'} \colon \Gamma'_{U'} \to \mathrm{Homeo(U')}\}$$

The map $\Phi_U \colon \Gamma_U \to \mathcal{H}_U$ may have kernel, and likewise for $\Phi'_{U'}$.

That is, return equivalence loses some of the information about the group action, and it is necessary to control this loss.

**Definition:** Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer action, and $\widehat{x} \in \mathfrak{X}$.

• A *dynamical commensurator* is a homeomorphism $h_U : U \to V$, where $U$ and $V$ are adapted subsets with $\widehat{x} \in U \cap V$, such that $h_U(\widehat{x}) = \widehat{x}$, and $h_U$ induces an isomorphism $\Theta_U : \mathcal{H}_U \to \mathcal{H}_V$.

• Commensurators $h_U : U \to V$ and $h_{U'} : U' \to V'$ are equivalent if there exists and adapted set $U''$ with $\widehat{x} \in U'' \subset U \cap U'$ such that $h_U | U'' = h'_{U'} | U''$. We write $(h_U, U, V) \overset{c}{\sim} (h'_{U'}, U', V')$.

The *dynamical commensurator group* of $(\mathfrak{X}, \Gamma, \Phi)$ at $\widehat{x}$ is the set of germs of dynamical commensurators,

$$\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, \widehat{x}) = \{h : U \to V \mid \widehat{x} \in U \cap V\} / \overset{c}{\sim}$$

The dynamical commensurator group is implicitly studied in

⋆ Hurder & Lukina, *Limit group invariants for non-free Cantor actions*, **Ergodic Theory Dynam. Systems**, 41:1751-1794, 2021.

Recall a result on classifying weak solenoids from

⋆ Clark, Hurder & Lukina, *Classifying matchbox manifolds*, *Geom. Topol.*, 23(1):1–27, 2019.

**Theorem.** Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are weak solenoids. If $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic, then their monodromy odometers $\Phi\colon \Gamma \times X_{\mathcal{P}} \to X_{\mathcal{P}}$ and $\Phi'\colon \Gamma' \times X'_{\mathcal{P}'} \to X'_{\mathcal{P}'}$ are *return equivalent*.

**Corollary:** For $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ there is a well-defined map

$$\sigma_{\mathcal{P},\widehat{x}}\colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \longrightarrow \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x})$$

To understand the image of this map, the strategy is to construct an embedding $\chi_{\Phi}\colon \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \longrightarrow \mathrm{Comm}(\Gamma)$.

Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an odometer. The action is:

- *quasi-analytic* if for each clopen set $U \subset \mathfrak{X}$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on $U$, then $\Phi(g)$ acts as the identity on $\mathfrak{X}$.
- *topologically free* if it is effective, and the quasi-analytic condition holds for $U = \mathfrak{X}$.
- *locally quasi-analytic* if there exists $\epsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$, and for any non-empty open subset $V \subset U$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on $V$, then $\Phi(g)$ acts as the identity on $U$.
- *coherent* if for every adapted set $U \subset \mathfrak{X}$, the restricted holonomy action map $\Phi_U\colon \Gamma_U \to \mathit{Homeo}(U)$ has finite kernel.

**Example:** If $\Gamma$ is residually finite, and $\widehat{\Gamma}$ is any profinite completion for which the canonical map $\Gamma \to \widehat{\Gamma}$ is an embedding, then the odometer action of $\Gamma$ on $\mathfrak{X} = \widehat{\Gamma}$ is coherent and free.

**Example:** If $\Gamma$ is virtually nilpotent, that is, $\Gamma$ contains a nilpotent subgroup $\Gamma' \subset \Gamma$ with finite index, then an odometer action of $\Gamma$ is coherent and locally quasi-analytic.

**Example:** If $\Gamma$ is a weakly branch group, and $\mathfrak{X}$ is the boundary of the tree on which the group acts, then the odometer action of $\Gamma$ on $\mathfrak{X}$ is inherently not coherent, and not locally quasi-analytic.

**Proposition:** Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective and coherent odometer action, and $U \subset \mathfrak{X}$ an adapted set. Then there exists a subgroup $\Gamma'_U \subset \Gamma_U$ of finite index in $\Gamma$ such that the restriction $\Phi_U\colon \Gamma'_U \to \mathrm{Homeo(U)}$ is injective.

*Proof.* The action map $\Phi\colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ is injective.

$\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$ is subgroup of finite index in $\Gamma$.

$K_U = \ker\{\Phi_U\colon \Gamma_U \to \mathrm{Homeo(U)}\} \subset \Gamma$ is a finite group.

Let $U' \subset U$ be adapted and sufficiently small such that $K_U$ acts effectively on the finite set of translates $X_U = \{\gamma \cdot U' \mid \gamma \in \Gamma\}$.

Then $\Gamma'_U = \bigcap_{\gamma \in \Gamma} \Gamma_{\gamma \cdot U'}$ has finite index in $\Gamma_U$ hence also in $\Gamma$.

$\Gamma'_U$ acts trivially on the set of translates $X_{U'}$. It follows that the restriction $\Phi_U\colon \Gamma'_U \to \mathrm{Homeo(U)}$ is injective. □

**Theorem:** Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective, coherent and locally quasi-analytic odometer. Let $\widehat{x} \in \mathfrak{X}$. Then there exists an injection

$$\chi_\Phi \colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, \widehat{x}) \longrightarrow \mathrm{Comm}(\Gamma)$$

*Proof.* Let $h\colon U \to V$ with $h(\widehat{x}) = \widehat{x}$, and $h_*$ conjugates the images $\mathcal{H}_U = \Phi_U(\Gamma_U) \subset \mathrm{Homeo(U)}$ with $\mathcal{H}_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo(V)}$.

Let $\Gamma'_U \subset \Gamma_U$ be finite index and $\Phi_U \colon \Gamma'_U \to \mathrm{Homeo(U)}$ is injective

Let $\Gamma'_V \subset \Gamma_V$ be finite index and $\Phi_V \colon \Gamma'_V \to \mathrm{Homeo(V)}$ is injective

Then $\Gamma''_U = \Phi_U^{-1}\{\Phi_U(\Gamma'_U) \cap h_*^{-1}(\Phi_V(\Gamma'_V))\}$ has finite index in $\Gamma$

Then $\varphi = \Phi_V^{-1} \circ h_* \circ \Phi_U \colon \Gamma''_U \to \Gamma''_V$ is a commensurator.

Set $\chi_\Phi[h, U, V] = \varphi$. $\square$

A finite *CW*-complex $Y$ is *aspherical* if it is connected and its universal covering space is contractible. Equivalently, $Y$ is aspherical if all homotopy groups $\pi_\ell(Y, y_0) = 0$ for $\ell > 1$.

A manifold $M$ is *Borel* that if it is a closed aspherical manifold, and a homotopy self-equivalence is homotopic to a self-homeomorphism. We require a somewhat stronger condition:

**Definition:** A closed connected manifold $M$ is said to be *strongly Borel* if every finite covering of $M$ is a Borel manifold.

The following classes of closed manifolds are strongly Borel:

- infra-nilmanifolds,
- closed Riemannian manifolds $M$ with negative sectional curvatures,
- closed Riemannian manifolds $M$ of dimension $n \neq 3, 4$ with non-positive sectional curvatures.

We combine the above results to obtain a method to analyze $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ when $M_0$ is a strongly Borel manifold. We require another result from

⋆ CHL, *Classifying matchbox manifolds*, *Geom. Topol.*, 23(1):1–27, 2019.

**Theorem.** Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are weak solenoids, and

1) $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ have the same dimension,
2) their monodromy actions are return equivalent,
3) the base manifolds $M_0$ and $M'_0$ are *strongly Borel*,
4) each space contains a simply connected leaf;

then $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic.

Above results applied to the self-homeomorphisms of $\mathfrak{M}_{\mathcal{P}}$ yields:

**Theorem:** Let $\mathcal{P}$ be a presentation such that $M_0$ is strongly Borel, and the associated odometer $(X_{\mathcal{P}}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Then for $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ we have the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \xrightarrow{\chi_{\Phi}} \mathrm{Comm}(\Gamma)$$

where

- $\sigma_{\mathcal{P},\widehat{x}}$ is surjective
- $\chi_{\Phi}$ is injective

The next step is to analyze the map $\chi_{\Phi}$ in examples.

The case when $M_0 = \mathbb{S}^1$ is classical. Note that $\Gamma = \mathbb{Z}$.

It is an exercise that $\mathrm{Comm}(\mathbb{Z}) \cong \mathbb{Q}^*$.

**Example 1:** Let $\mathcal{P}$ be given by the maps $p_\ell(z) = z^{\ell+1}$, for $\ell \geq 0$.

$p_\ell$ is covering corresponding to subgroup $(\ell + 1) \cdot \mathbb{Z} \subset \mathbb{Z}$.

The fiber $\mathfrak{X}$ of $\widehat{q}_0 \colon \mathfrak{M}_\mathcal{P} \to \mathbb{S}^1$ is the full profinite completion $\widehat{\mathbb{Z}}$

$\mathfrak{M}_\mathcal{P}$ is the *universal solenoid* over $\mathbb{S}^1$.

We can calculate $\mathrm{Mod}(\mathfrak{M}_\mathcal{P})$ directly.

For $m/n \in \mathbb{Q}^*$ define $\varphi_{m/n}(x) = \frac{m}{n} \cdot x$ which induces
$\overline{\varphi}_{m/n} \colon M_n = \mathbb{R}/n\mathbb{Z} \to M_m = \mathbb{R}/m\mathbb{Z}$ and $h_{m/n} \colon \mathfrak{M}_\mathcal{P} \to \mathfrak{M}_\mathcal{P}$

For any $\widehat{z} \in \widehat{\mathbb{Z}}$ also have map $\widehat{z} \colon \mathfrak{M}_\mathcal{P} \to \mathfrak{M}_\mathcal{P}$ where $\widehat{z}(y) = y \cdot \widehat{z}$

**Theorem:** $\mathrm{Mod}(\mathfrak{M}_\mathcal{P}) \cong \mathbb{Q}^* \times \widehat{\mathbb{Z}}$ , $\mathrm{Mod}(\mathfrak{M}_\mathcal{P}, \widehat{x}) \cong \mathbb{Q}^*$

**Example 2:** Let $\mathcal{P}$ be coverings defined by the chain of subgroups

$$\Gamma = \mathbb{Z} \supset \Gamma_1 = m_1 \cdot \mathbb{Z} \supset \Gamma_2 = m_1 m_2 \cdot \mathbb{Z} \supset \Gamma_3 = m_1 m_2 m_3 \cdot \mathbb{Z} \supset \cdots$$

where each $m_i > 1$. Set $\vec{m} = \{m_1, m_2, m_3, \ldots\}$, $X_{\mathcal{P}} = X_{\vec{m}}$.

The odometer $(X_{\vec{m}}, \mathbb{Z}, \Phi)$ is free, and $X_{\vec{m}}$ is a profinite completion of $\mathbb{Z}$. We then have, for any $\widehat{x} \in X_{\vec{m}}$:

- $\sigma_{\mathcal{P},\widehat{x}} \colon \mathrm{Mod}(\mathfrak{M}_{\vec{m}}, \widehat{x}) \to \mathrm{Comm}(X_{\vec{m}}, \mathbb{Z}, \Phi, \widehat{x})$ is an isomorphism
- $\chi_\Phi \colon \mathrm{Comm}(X_{\vec{m}}, \mathbb{Z}, \Phi, \widehat{x}) \to \mathrm{Comm}(\mathbb{Z})$ is an inclusion.

Almost everything to say about these solenoids is well-known: see

⋆ Kwapisz, *Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua*, **Fund. Math.**, 168(3):251–278, 2001

and for entropy calculations, see:

⋆ Lind & Ward, *Automorphisms of solenoids and p-adic entropy*, **Ergodic Theory Dynam. Systems**, 8:411–419, 1988.

**Detour:** Given $\vec{m} = \{m_1, m_2, m_3, \ldots\}$, the *supernatural number* (or *Steinitz number*) defined by $\vec{m}$ is:

$$\xi(\vec{m}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} \ ,$$

$\mathrm{lcm}$ denotes the least common multiple of the collection of integers. A Steinitz number $\xi$ can be written uniquely as the formal product over the set of primes $\Pi$,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)}$$

The *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime $p$ appears in the infinite product $\xi$.

Two Steinitz numbers $\xi$ and $\xi'$ are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$

The *type* associated to a Steinitz number $\xi$ is the asymptotic equivalence class of $\xi$, denoted by $\tau[\xi]$.

**Lemma.** $\xi$ and $\xi'$ satisfy $\xi \overset{a}{\sim} \xi'$ if and only if their characteristic functions $\chi, \chi'$ satisfy

- $\chi(p) = \chi'(p)$ for all but finitely many primes $p \in \Pi$
- $\chi(p) = \infty$ if and only iff $\chi'(p) = \infty$ for all primes $p \in \Pi$.

Given $\xi = \prod_{p \in \Pi} p^{\chi(p)}$, define:

$$\begin{aligned} \pi(\xi) &= \{p \in \Pi \mid \chi(p) > 0\}\ , \ \textit{prime spectrum of } \xi \\ \pi_f(\xi) &= \{p \in \Pi \mid 0 < \chi(p) < \infty\}\ , \ \textit{finite prime spectrum of } \xi \\ \pi_\infty(\xi) &= \{p \in \Pi \mid \chi(p) = \infty\}\ , \ \textit{infinite prime spectrum of } \xi \end{aligned}$$

**End of detour.**

Now return to $\chi_\Phi \colon \mathrm{Comm}(X_{\vec{m}}, \mathbb{Z}, \Phi, \widehat{x}) \to \mathrm{Comm}(\mathbb{Z}) \cong \mathbb{Q}^*$

**Proposition:** Let $\xi$ be the type of $\vec{m}$. Then multiplication by $p$
$\times_p \in \mathrm{Image}(\chi_\Phi)$ if and only if $p \in \pi_\infty(\xi)$

**Theorem:** Let $\mathcal{P}$ be a presentation such that $M_0 = \Sigma_g$ for $g \geq 2$. Assume that the associated odometer $(\mathfrak{X}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Consider the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \xrightarrow{\chi_\Phi} \mathrm{Comm}(\Gamma)$$

The map $\sigma_{\mathcal{P},\widehat{x}}$ is an isomorphism, and the map $\chi_\Phi$ is injective.

When $\mathcal{P}$ is the tower of all finite coverings of $\Sigma_g$, the map $\chi_\Phi \circ \sigma_{\mathcal{P},\widehat{x}} \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$ is an isomorphism, and so this recovers the result of Odden.

**Problem:** For other presentations $\mathcal{P}$ with base $\Sigma_g$ calculate the image of $\chi_\Phi \colon \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \to \mathrm{Comm}(\Gamma)$.

- The case when $M_0 = \mathbb{T}^2$ is an interesting exercise.

**Theorem:** Let $M_0$ be a closed strongly Borel manifold. Assume that $\Gamma = \pi_1(M_0, x)$ is residually finite. Let $\mathcal{P}$ be a presentation such that the associated odometer $(\mathfrak{X}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Consider the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \xrightarrow{\chi_{\Phi}} \mathrm{Comm}(\Gamma)$$

The map $\sigma_{\mathcal{P},\widehat{x}}$ is a surjection, and the map $\chi_{\Phi}$ is injective.

**Remark:** The problem is thus to calculate the image of $\chi_{\Phi}$. If the transversal $X_{\mathcal{P}}$ is a profinite group, this calculation is independent of the choice of basepoint $\widehat{x}$. Otherwise, the calculation may depend on the choice of $\widehat{x}$, which determines the group chain $\mathcal{G}_{\mathcal{P}}$.

We then have the following extension of the results of Belk & Forrest, and Edgar & Studenmund:

**Theorem:** Let $M_0$ be a closed strongly Borel manifold. Assume that $\mathcal{P}$ is a presentation such that the action $(X_{\mathcal{P}}, \Gamma, \Phi)$ is effective, and $\mathfrak{M}_{\mathcal{P}}$ is a normal solenoid. Then

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}) \cong \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \times \widehat{\Gamma}_{\mathcal{P}}$$

and for the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \xrightarrow{\chi_{\Phi}} \mathrm{Comm}(\Gamma)$$

the map $\sigma_{\mathcal{P},\widehat{x}}$ is a surjection, and the map $\chi_{\Phi}$ is injective.

Let $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \ , \ \ell \geq k\}$ be a presentation, and let $m_\ell = \deg(p_{\ell+1})$. Let $\vec{m} = \{m_1, m_2, m_3, \ldots\}$ and let $\tau(\mathcal{P})$ be the type of the Steinitz number $\xi(\mathcal{P})$ associated to $\vec{m}$.

**Theorem:** Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are presentations such that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic. Then $\tau(\mathcal{P}) = \tau(\mathcal{P}')$.

$\star$ Hurder & Lukina, *Type invariants for solenoidal manifolds*, **Ergodic Theory Dynam. Systems**, 2025.

**Problem:** Let $M_0$ be a closed strongly Borel manifold, with $\Gamma = \pi_1(M_0, x)$ residually finite. For which types $\tau$

- does there exists $\mathcal{P}$ with base $M_0$ such that $\tau(\mathcal{P}) = \tau$?
- is the map $\chi_\Phi \colon \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \to \mathrm{Comm}(\Gamma)$ injective?

The case when $M_0$ is a nilmanifold is treated in the papers

⋆ Hurder & Lukina, *Mapping class groups of solenoidal manifolds*, in preparation, 2025.

⋆ Hurder & Lukina, *Mapping class groups of nilpotent solenoidal manifolds*, in preparation, 2025.