Foliation index theory for weak solenoids

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Joint work with Olga Lukina
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[1983] A long time ago...

The Mathematical Institute,
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30th July 1983

Dear Dr. Hurder,

Thank you for your letter of 1 July, which was waiting for me when I returned to Oxford a few days ago.

I'm afraid that I have nothing which is in a sufficiently final form for me to send to you, but I will try to describe the idea of my work. The 'grand design' is to mimic Connes procedure for certain non compact Riemannian manifolds instead of compact manifolds. The manifolds I consider are those with "bounded geometry" which means a lower bound on the injectivity radius and an upper bound on the curvature tensor and its covariant derivatives. Such manifolds, for example, occur as the leaves of foliations of compact manifolds - in fact it seems to be an open question whether they all do. Now to such a Riemannian manifold (M,g) I propose to associate a C^* algebra $C^*(M,g)$ in such a way that the corresponding K-theory group $K_0(C^*(M,g))$ will contain the indices of the 'natural' elliptic operators on M such as the de Rham operator $d+d^*$. Next, some manifolds (in fact exactly those manifolds "closed at infinity" in the sense of Sullivan, Inventiones 36) admit certain functionals which I call 'invariant means'; these play the role of transverse measures in the Connes theory, giving rise to ~~x~~ dimension functions $K_0(C^*(M,g)) \to \mathbb{R}$. There seems to be a Gauss Bonnet formula ~~with~~ which expresses the real valued index of the de Rham operator in terms of curvature - modulo some analysis which is proving rather recalcitrant at present. Finally, I hope to be able to compute $K_0(C^*(M,g))$ for nice M - e.g. symmetric spaces or deformations thereof - by means of representation theory.

If you think this is interesting let me know and I will try and let you have a copy of anything respectable.

Yours sincerely,

John Roe

John Roe (Mr)

PS. One interesting aspect of these~~xindx~~ results is that they seem to be closely related - at least in the 2 dimensional case - to classical Nevanlinna theory.


John Roe

Problem [Folklore 1974]: Geometry of leaves of foliations.

Problem [Atiyah 1986]: Index theory for coverings.

Problem [Connes 1980]: Index theory of foliations.

Problem [Roe 1983]: Index theory of complete open manifolds.

Goal of Talk: Weak solenoids = class of foliated spaces whose leaf geometries remain unclassified.

Goal of Program

- Structure of weak solenoids.
- Index Theory for weak solenoids.
- Spectral geometry of leaves in weak solenoids
- Invariants from C^*-algebras associated to Cantor actions.

⋆ Spectral flow, Lück approximation for irregular coverings.

⋆ Bowdoin Lecture [1988] on η-invariants and spectral flow.

- M is compact manifold without boundary
- $G = \pi_1(M, x_0)$ is finitely generated group.

$$M = M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} M_3 \cdots$$

Choose $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(M_\ell, x_\ell)$

Inclusion maps $q_\ell \colon G_\ell \subset G_{\ell-1}$, descending chain of groups

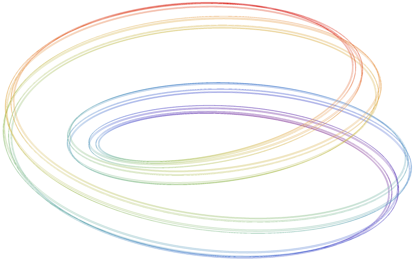
$$G = G_0 \xleftarrow{q_1} G_1 \xleftarrow{q_2} G_2 \xleftarrow{q_3} G_3 \cdots$$

Tower of coverings is *normal* if each $G_\ell \subset G_0$ is a normal subgroup.

Example: Vietoris solenoid is given by chain of subgroups of $\mathbb{Z}$

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \mathbb{S}^1 \cdots$$

- ⋆ each $m_\ell > 1$ is an integer
- ⋆ $m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1$ is an m_ℓ-fold covering map.



Inverse limit space = Vietoris solenoid [1927]

$$\mathcal{S} \equiv \varprojlim \{ m_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 0\}$$

$$\equiv \{(x_0, x_1, \ldots) \in \mathcal{S} \mid m_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0 \}$$

$$\subset \prod_{\ell \geq 0} \mathbb{S}^1$$

- ⋆ Give the product space the product or Tychonoff topology;
- ⋆ $\mathcal{S}$ has the restricted topology;
- ⋆ $\mathcal{S}$ is a compact, connected metric space, so a continuum.

Properties of the space $\mathcal{S}$

$\star$ $\mathcal{S}$ is an uncountable union of lines, each dense in $\mathcal{S}$.

$\star$ Whatever stage $k \geq 0$ you start at, the limit is the same:

$$\mathcal{S} \cong \varprojlim \{ m_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq k \}$$

$\star$ $\mathcal{S}$ is homogeneous - for any pair of points $x, y \in \mathcal{S}$ there is a homeomorphism $h \colon \mathcal{S} \to \mathcal{S}$ with $h(x) = y$.

- R.H. Bing, Canad. J. Math., 1960
- M.C. McCord, Transactions A.M.S, 1965

Theorem: Let $\vec{p} = \{p_1, p_2, p_3, \ldots\}$ and $\vec{q} = \{q_1, q_2, q_3, \ldots\}$ be sequences of prime numbers. Say that $\vec{p} \sim \vec{q}$ if there is $\ell_p, \ell_q \geq 0$ so that the two sets

$$\{p_\ell \mid \ell \geq \ell_p\} = \{q_\ell \mid \ell \geq \ell_q\}$$

Then the solenoids they generate, call them $\mathcal{S}(\vec{p})$ and $\mathcal{S}(\vec{q})$, are homeomorphic if and only if $\vec{p} \sim \vec{q}$.

Question: Is there a similar result in higher dimensions?

Inverse limit space for a tower of coverings:

$$\begin{aligned} M_\infty = \varprojlim \; \{p^\ell_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \\ = \; \{(y_0, y_1, y_2, \ldots) \mid p^\ell_{\ell+1}(y_{\ell+1}) = y_\ell \mid \ell \geq 0\} \\ \subset \; \prod_{\ell \geq 0} M_\ell \end{aligned}$$

is a compact connected metrizable space called a *(weak) solenoid*.

For each $\ell > 0$, there is a fibration map $\Pi_\ell \colon M_\infty \to M_\ell$.

For fixed $x_\ell \in M_\ell$ the fiber $\mathfrak{X}_\ell = \Pi_\ell^{-1}(x_\ell) \subset \mathfrak{X}_0$ is a Cantor space.

The path connected components of M_∞ are manifolds, which are non-compact covering spaces for M_0, so that M_∞ is a generalized lamination, or foliated space with Cantor transversals.

The fundamental group $G_0 = \pi_1(M_0, x_0)$ acts on the fiber $\mathfrak{X}_0$ via lifts of paths in M_0 to the leaves of $\mathcal{F}_{\mathfrak{M}}$ giving the monodromy action on the fiber, $\Phi \colon G_0 \times \mathfrak{X}_0 \to \mathfrak{X}_0$. The action is minimal, the orbit of each point is dense in $\mathfrak{X}_0$.

Proposition: The monodromy is an *equicontinuous group action*.

- A Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is <u>equicontinuous</u> if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in G.$$

When $G = \mathbb{Z}$, then a minimal equicontinuous Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is conjugate to a classical odometer.

Cantor action $(\mathfrak{X}, G, \Phi)$ is minimal equicontinuous action.

Problem: Classify Cantor actions up to topological conjugacy, or continuous orbit equivalence, or return equivalence.

Definition: $\mathfrak{G}(\Phi) = \overline{H_\Phi}$ = closure of $H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X})$ in the *uniform topology on maps*. $\mathfrak{G}(\Phi)$ is profinite group.

For $x \in \mathfrak{X}$, $\mathcal{D}_x = \{\widehat{h} \in \overline{H_\Phi} \mid \widehat{h} \cdot x = x\} = \mathfrak{G}(\Phi)_x$ (isotropy group)

Lemma: Transitive left action of $\mathfrak{G}(\Phi)$ on $\mathfrak{X} \cong \mathfrak{G}(\Phi)/\mathcal{D}_x$.

Closed subgroup $\mathcal{D}_x \subset \mathfrak{G}(\Phi)$ is independent of the choice of basepoint x, up to topological isomorphism.

The action $(\mathfrak{X}, \mathfrak{G}(\Phi), \widehat{\Phi})$ is called the profinite model for $(\mathfrak{X}, G, \Phi)$.

$$\begin{array}{ccc} & & \mathfrak{G}(\Phi) \\ & \overset{\widehat{\Phi}}{\nearrow} & \downarrow \\ G = \pi_1(M_0, x_0) & \xrightarrow{\Phi} & \mathfrak{X} \end{array}$$

Theorem: $\mathcal{D}_x$ is trivial if and only if $\mathfrak{X}$ is a Cantor group.

So for $\mathcal{D}_x$ trivial, we have a tower of normal coverings, as in Cheeger-Gromov [1985], Lück [1994], ...

On the other hand, the case when $\mathcal{D}_x$ is non-trivial seems to be almost completely unknown in the literature.

Problem: Relate asymptotic geometry & analysis of leaves in weak solenoids to algebraic properties of subgroup $\mathcal{D}_x \subset \mathfrak{G}(\Phi)$.

Proposition: $\mathcal{D}_x$ is *totally not normal*: for any $\widehat{h} \in \mathcal{D}_x$ there exists $\widehat{g} \in \mathfrak{G}(\Phi)$ such that $\widehat{g}^{-1}\ \widehat{h}\ \widehat{g} \notin \mathcal{D}_x$.

For group chain $\mathcal{G} = \{G_\ell \mid \ell \geq 0\}$ the normal core of G_ℓ in G

$$C_\ell = \bigcap_{g \in G} g G_\ell g^{-1} \subset G_\ell$$

Theorem [Dyer-Hurder-Lukina, 2016].

$$\mathcal{D}_x \equiv \varprojlim \ \{\pi_{\ell+1} \colon G_{\ell+1}/C_{\ell+1} \to G_\ell/C_\ell \mid \ell \geq 0\} \ .$$

1. $\mathcal{D}_x$ is trivial for Cantor action $(\mathfrak{X}, G, \Phi)$ with G abelian.
2. $\mathcal{D}_x$ can be a Cantor group for a Cantor action $(\mathfrak{X}, G, \Phi)$ when G is 3-dimensional Heisenberg group.
3. Every finite group and every separable profinite group can be realized as $\mathcal{D}_x$ for a Cantor action by a torsion-free, finite index subgroup of $\mathbf{SL}(n, \mathbb{Z})$.
4. $\mathcal{D}_x$ can be wide-ranging for arboreal representations of absolute Galois groups of number fields and function fields.
5. Every Cantor action by a finitely generated group G can be realized by a tower of finite coverings of a closed surface.

The proof of 3 uses ideas of Lubotzky [1993] on torsion elements in the profinite completion of torsion free subgroups of $\mathbf{SL}(n, \mathbb{Z})$, and a construction due to **Lenstra**.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is either stable or wild.

Choose a neighborhood basis $\{\widehat{C}_\ell \mid \ell \geq 1\}$ of the identity $\widehat{e} \in \mathfrak{G}(\Phi)$. Each $\widehat{C}_\ell$ is a normal subgroup of $\mathfrak{G}(\Phi)$.

Nested: $\widehat{C}_{\ell+1} \subset \widehat{C}_\ell$, with $\cap_{\ell \geq 1} \widehat{C}_\ell = \{\widehat{e}\}$.

Given $\mathcal{D}_x \subset \mathfrak{G}(\Phi)$, define $\widehat{U}_\ell = \mathcal{D}_x \cdot \widehat{C}_\ell$. Then $U_\ell = \widehat{U}_\ell / \mathcal{D}_x$ is clopen neighborhood of basepoint $x \in \mathfrak{X}$.

$\Phi(G) \subset \mathfrak{G}(\Phi)$ is dense subgroup; $G_\ell = \{g \in G \mid \Phi(g) \in \widehat{U}_\ell\}$.

$\Phi_\ell \colon G_\ell \times U_\ell \to U_\ell$ is minimal action; $\widehat{\Phi}_\ell \colon \widehat{G}_\ell \times U_\ell \to U_\ell$.

$\mathcal{D}_x \in \widehat{U}_\ell$ for all $\ell \geq 1$, $\mathcal{D}_x = \cap_{\ell \geq 1} \widehat{U}_\ell$.

$\widehat{\Phi}_\ell$ induces map $\rho_\ell \colon \mathcal{D}_x \to \mathbf{Homeo}(U_\ell)$;

Set $K_\ell \equiv \ker\{\rho_\ell\}$ for $\ell \geq 1$. Then $K_1 \subset K_2 \subset \cdots$

Theorem: The isomorphism class of the direct limit group

$$\Upsilon(\Phi) = \varinjlim \, \{K_\ell \subset K_{\ell+1} \mid \ell \geq 1\}$$

is a conjugacy invariant of a Cantor action $(\mathfrak{X}, G, \Phi)$.

A Cantor action $(\mathfrak{X}, G, \Phi)$ is:

- <u>stable</u> if the chain $\{K_\ell \mid \ell \geq 1\}$ is bounded.

That is, if there exists ℓ_0 so that $K_\ell = K_{\ell+1}$ for $\ell \geq \ell_0$.

- <u>wild</u> if the chain $\{K_\ell \mid \ell \geq 1\}$ is unbounded.

Theorem [Hurder-Lukina 2019]: The property that a Cantor action is wild is invariant by a continuous orbit equivalence.

$\star$ The examples of group actions on trees generated by automata studied by Nekrashevych, Bartholdi, Grigorchuk *et al* typically induce wild actions on the Cantor boundary of a tree.

Theorem [Lukina 2018]: Let p and d be distinct odd primes, let $K = \mathbb{Q}_p$ be the field of p-adic numbers. Let $f(x) = (x + p)^d - p$. Then the action of $\mathrm{Gal}_\infty(f)$ is stable.

Theorem [Lukina 2018]: Let $f(x)$ be a quadratic polynomial with critical point c. If the post-critical set P_C is infinite, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$, and so of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.

There is a geometric interpretation of the stable/wild property.

Definition: A topological action $\Phi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is *locally quasi-analytic (LQA)* if there exists $\epsilon > 0$ such that for any open set $U \subset \mathfrak{X}$ with $diam(U) < \epsilon$, and for any open $V \subset U$ and $g_1, g_2 \in G$ if

$$\text{if } \Phi(g_1)|V = \Phi(g_2)|V \text{ then } \Phi(g_1)|U = \Phi(g_2)|U \ .$$

Alternatively, the action is locally quasi-analytic if and only if for all $g \in G$ if $\Phi(g)|V = id$, then $\Phi(g)|U = id$, for open sets $V \subset U$.

Theorem [Hurder-Lukina, 2017]: A Cantor action $(\mathfrak{X}, G, \Phi)$ with G finitely generated is stable, if and only if the action $\widehat{\Phi}\colon \mathfrak{G}(\Phi) \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic.

Let $\mathfrak{X}$ be a Cantor set, and let $\mathcal{H} \subset \mathbf{Homeo}(\mathfrak{X})$ be a group.

An element $h \in \mathcal{H}$ is *non-Hausdorff* if there exists $x \in \mathfrak{X}$ and a collection of open sets $\{U_n\}_{n \geq 1}$ with $\bigcap U_n = \{x\}$, such that

1. $h(x) = x$,
2. $h|_{U_n}$ is non-trivial,
3. for $n \geq 1$, there exists an open set $O_n \subset U_n$ with $h|_{O_n} = id$.

Theorem: [Winkelnkemper, 1983] The germinal étale groupoid $\Gamma(\mathfrak{X}, G, \Phi)$ associated to a Cantor action $(\mathfrak{X}, G, \Phi)$ is non-Hausdorff if and only if $\mathcal{H} = \Phi(G)$ contains a non-Hausdorff element.

Definition: A Cantor action $(\mathfrak{X}, G, \Phi)$ has finite type if each group in the chain $\{K_\ell \mid \ell \geq 1\}$ of isotropy groups is finite.

Theorem [Hurder-Lukina, 2019]: $(\mathfrak{X}, G, \Phi)$ a Cantor action:

- If $\mathfrak{G}(\Phi)$ contains a non-Hausdorff element $\Rightarrow$ action is wild.
- G is finitely generated and action is wild of finite type $\Rightarrow$ $\mathfrak{G}(\Phi)$ contains a non-Hausdorff element.

Example [Lukina 2018]: This is false if we only assume that G is a countable group. There are examples of absolute Galois groups for function fields, whose arboreal representations are wild, but do not have a non-Hausdorff element.

Here are two further results:

Theorem [Hurder-Lukina 2018]: There exists uncountably many wild actions of torsion-free finite index subgroups of $\mathbf{SL}(n, \mathbb{Z})$ with distinct asymptotic discriminants.

Theorem [Hurder-Lukina 2018]: Let $\Phi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be a Cantor action with G a finitely generated nilpotent group. Then the action is stable. Moreover, any Cantor action which is continuously orbit equivalent must be return equivalent. That is, their germinal étale groupoids are Morita equivalent.

Let $C_r^*(\mathfrak{X}, G, \Phi)$ be the reduced C^*-algebra associated to the Cantor action $(\mathfrak{X}, G, \Phi)$.

Let $\mathfrak{M}$ be a weak solenoid whose monodromy action is return equivalent to the Cantor action $(\mathfrak{X}, G, \Phi)$.

Let $\mathcal{F}$ be the foliation on $\mathfrak{M}$, and let D be a leafwise elliptic operator along the leaves of $\mathcal{F}$.

Then D induces a KK-class $[D] \in KK(C(\mathfrak{M}), C_r^*(\mathfrak{X}, G, \Phi))$.

Let $\mathfrak{X} = \mathfrak{G}(\Phi)/\mathcal{D}_x$ be the profinite model for the action.

A representation $\rho\colon \mathcal{D}_x \to U(N)$ gives $[\rho] \in KK^1(\mathbb{C}, C(\mathfrak{M}))$

Form the index pairing

$$[\rho]\colon KK(C(\mathfrak{M}), C_r^*(\mathfrak{X}, G, \Phi)) \longrightarrow K_*(C_r^*(\mathfrak{X}, G, \Phi))$$

Question: Do the values of the index pairing suffice for showing that a Cantor action $(\mathfrak{X}, G, \Phi)$ is wild?

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Thank you for your attention!

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