Cohomology and smooth embeddings for matchbox manifolds[1]

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[1] Matchbox Dynamics - Mini-Symposium

Topics

- Matchbox manifolds – smooth foliated spaces $\mathfrak{M}$ with codimension zero.
- Smooth foliated embeddings of $\mathfrak{M}$ into smooth foliated manifolds M.
- Shape and the *fixed-point property*.
- *Non-commutative* self-linking invariants.

Matchbox manifolds

Let $\mathfrak{M}$ be a continuum – a nonempty compact connected metric space.

Definition: $\mathfrak{M}$ is an n-*dimensional matchbox manifold* $\Longleftrightarrow$ $\mathfrak{M}$ admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$ where $\mathfrak{T}_i$ is a totally disconnected *clopen* subset of a Polish space $\mathfrak{X}$.

Generalizes concept of continuum with every proper subcontinuum an arc, and arc-components are leaves of a foliation $\mathcal{F}_{\mathfrak{M}}$, where local charts are "matchboxes".

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The transition functions are assumed to be C^r, for $1 \leq r \leq \infty$, along leaves, and the derivatives depend (uniformly) continuously on the transverse parameter.

$\mathfrak{M}$ is a smooth foliated space with codimension zero and leaf dimension n.

Similar concept to laminations, but not necessarily 1 or 2 dimensional, and not given as embedded subspace.

$\mathfrak{M}$ denotes a smooth matchbox manifold.

• Arise in study of tiling spaces, dynamics of foliations, graph constructions, inverse limit spaces, pseudogroup actions on Cantor spaces, et cetera.

• Analogous to a compact manifold, with unique quasi-isometry class of Riemannian metrics on the leaves of $\mathcal{F}_{\mathfrak{M}}$.

• The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ for $\mathfrak{M}$, acting on a complete (Cantor) transversal $\mathfrak{X}$, yields a well-defined dynamical system associated to $\mathfrak{M}$.

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Study of matchbox manifolds has been active for fifty years at least in various contexts. The current research program looks at the overall picture for this class of spaces. In some ways, it is analogous to study of properties of embedding Riemannian manifolds $L \subset M$, though with much greater complexity.

Intrinsic properties

Study $\mathfrak{M}$ in terms of its *intrinsic* and *extrinsic* properties:

• Quasi-isometry class of leafwise metrics, yields "large scale" geometric invariant, such as growth properties of the leaves of $\mathcal{F}_{\mathfrak{M}}$ and foliation entropy $0 \leq h(\mathcal{G}_{\mathcal{F}}) \leq \infty$ as defined by Ghys-Langevin-Walczak.

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• [Clark & H, 2011] Action of $\mathcal{G}_{\mathcal{F}}$ on $\mathfrak{X}$ equicontinuous implies $\mathcal{F}_{\mathfrak{M}}$ is minimal, and $\mathfrak{M}$ is (foliated) homeomorphic to a generalized solenoid (described below).

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• Growth rates of Čech cohomology groups for $\mathfrak{M}$, as studied by Gambaudo & Martens for tiling spaces.

Invariant measures

$\mathfrak{X} \subset \mathfrak{M}$ is a compact transversal to $\mathcal{F}_{\mathfrak{M}}$ intersecting every leaf.

Definition: Probability measure μ on $\mathfrak{X}$ is invariant if for each $\phi \in \mathcal{G}_{\mathcal{F}}$ and compact Borel set $E \subset \text{Domain}(\phi)$, $\mu(E) = \mu(\phi(E))$.

• Exists if $\mathcal{G}_{\mathcal{F}}$ is equivalent to suspension of amenable group action on Cantor set.

• Exists if there is a leaf with subexponential volume growth, such as defined by a locally free $\mathbb{R}^n$-action, or G-action of a simply connected nilpotent Lie group G.

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Question: when does invariant measure μ for $\mathfrak{M}$ exist? and if so, can you describe the space of all such, in terms of topological of dynamical properties of $\mathfrak{M}$?

Extrinsic properties

Consider $\mathfrak{M}$ embedded as a closed invariant set of a smooth foliation for a (possibly open) foliated manifold M. The *intrinsic pseudogroup dynamics* reflected in *extrinsic properties of the embedding*.

• Invariant measure $\mu \Rightarrow$ codimension-one embedding must be Denjoy type.

• $f \colon \mathbf{K} \to \mathbf{K}$ a minimal action on Cantor set $\mathbf{K} \subset \mathbb{S}^1$, when does this action arise as the restriction of a C^r-diffeomorphism of $f \colon \mathbb{S}^1 \to \mathbb{S}^1$?

Denjoy Theorem implies $r < 2$, with restrictions on gap lengths for $1 \leq r < 2$.

Problem: Suppose given embedding of Cantor set $\mathbf{K} \subset \mathbb{R}^k$ for $k \geq 2$, when does a minimal action of finitely-generated group on $\mathbf{K}$ extend to a C^r-map of an open neighborhood of $\mathbf{K}$, for $r \geq 1$?

Fixed-Point Property (FPP)

Definition: The *shape* of $\mathfrak{M} \subset M$ is defined by a co-final descending chain $\{U_\ell \mid \ell \geq 1\}$ of open neighborhoods (called a shape approximation to $\mathfrak{M}$)

$$M \supset U_1 \supset U_2 \supset \cdots \supset U_\ell \supset \cdots \supset \mathfrak{M} \quad ; \quad \bigcap_{\ell=1}^{\infty} U_\ell = \mathfrak{M}$$

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Definition: A *foliated embedding* $\mathfrak{M} \subset M$ has the *fixed-point property* if there is a shape approximation $\{U_\ell \mid \ell \geq 1\}$ to $\mathfrak{M}$ so that for each $\ell \geq 1$ there is a *compact leaf* $L_\ell \subset U_\ell$ which is a homotopy retract.

The *fixed-point property* is extrinsic to $\mathfrak{M}$, though may depend on the embedding.

Remark: If one shape approximation to $\mathfrak{M} \subset M$ has the FPP, then it holds for all.

Problem: If $\mathfrak{M}$ has a C^r-foliated embedding $\mathfrak{M} \subset M$ with the FPP, must every C^r-foliated embedding have the FPP, for $r \geq 0$?

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Here is one application of the FPP:

Lemma: If $\mathfrak{M} \subset M$ has the *fixed-point property*, then there is invariant measure $\mu_{\mathcal{P}}$ on $\mathfrak{M}$ defined by compact approximations $\mathcal{P}$.

McCord solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\mathcal{P} \equiv \xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

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The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \lim_{\longleftarrow} \; \{p_\ell \colon B_\ell \to B_{\ell-1}\} \;\subset\; \prod_{\ell=0}^{\infty} \; B_\ell$$

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$\mathcal{S}$ is a *McCord solenoid* if every finite composition of coverings is normal.

If the base $B_0 = \mathbb{T}^n$ then every tower is a McCord solenoid.

Smooth embeddings

Theorem: Let $\mathcal{S}$ be a generalized solenoid over the base space $\mathbb{T}^n$, with presentation tower $\mathcal{P}$. Let $r \geq 2$ and let $q \geq 2n$. Subject to technical restrictions on the growth rates of the degree so the coverings in the tower, there exists a C^r-foliation $\widehat{\mathcal{F}}$ of $\mathbb{T}^n \times \mathbb{D}^q$ transverse to the disk fibers such that:

1. $\widehat{\mathcal{F}}$ is a distal foliation, with smooth transverse invariant volume form;
2. $L_0 = \mathbb{T}^n \times \{\vec{0}\}$ is a leaf of $\widehat{\mathcal{F}}$, and $\widehat{\mathcal{F}} = \mathcal{F}_0$ near the boundary of M;
3. the solenoid $\mathcal{S}$ embeds as a minimal set $\widehat{\mathcal{F}}$;
4. the embedding $\mathcal{S}$ has the FPP, realizing the compact manifolds $\{B_\ell \mid \ell \geq 1\}$ as leaves L_ℓ which are the homotopy retracts of the approximations to $\mathcal{S}$.

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Problem: For McCord solenoids, when is smooth embedding with FPP possible?

Self-linking invariants

Let $\vec{X}$ be non-vanishing vector field on Riemannian 3-manifold M.

Defines foliation $\mathcal{F}$ of codimension 2.

Assume there is closed invariant set $\mathfrak{M} \subset M$.

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Assume the normal bundle $Q = T\mathcal{F}^{\perp}$ restricted to open neighborhood $\mathfrak{M} \subset U \subset M$ is trivial: $TU \cong \langle \vec{X} \rangle \oplus Q$. $\Pi \colon TU \to Q$ orthogonal projection.

Define connection $\nabla_{\vec{X}}(\vec{Y}) = \Pi\left(L_{\vec{X}}\vec{Y}\right)$ for section $\vec{Y} \colon U \to Q$.

Extend to matrix-valued connection 1-form $\omega \colon TU \to \mathfrak{gl}(2, \mathbb{R})$.

Ruelle invariant

Choose "generic" point $x \in \mathfrak{M}$, so that μ is defined by averaging over orbit of x.

- Define an extrinsic invariant of $\mathfrak{M} \subset M$ (and the choice of μ and framing of Q)

$$R_\mu = \lim_{n \to \infty} \frac{1}{2n} \cdot \int_{-n}^{n} Tr\{\omega\}(\dot{\varphi}_t)\, dt \ \in \mathbb{R}$$

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If the support of μ is all of M, then R_μ equals the *Ruelle* invariant.

More generally, R_μ is the *first measured leaf class* for $\mathcal{F}$, as defined in

[Hurder] *Global invariants for measured foliations*, Trans. AMS 1983.

Non-commutative self-linking invariants

Definition: *Chern polynomials* $\mathcal{I}_q \equiv \mathbb{R}[c_1, \ldots, c_q]$ where $\deg(c_i) = 2i$.

Theorem: Let $\mathfrak{M}$ be embedded as a minimal set of a C^2-foliation $\mathcal{F}$ of a foliated manifold M, with leaves of dimension n and codimension $q \geq 2$.

Assume the the normal bundle $Q = T\mathcal{F}^{\perp}$ to the leaves of $\mathcal{F}$ in M is trivial, and the embedding has the FPP with tower $\mathcal{P} = \{L_\ell \mid \ell \geq 1\}$.

Then for each homogeneous polynomial $C \in \mathcal{I}_q$ with $2\ell = \deg(C) \geq 2$, there exists a well-defined *non-commutative self-linking invariant*, $\widehat{C}_{\mathcal{P}} \in \mathbb{R}$.

- For the case of a flow with $n = 1$, take $C = c_1$ then $d(\mathcal{P}) = 1$ and $\widehat{C}_{\mathcal{P}} = R_{\mu_{\mathcal{P}}}$.
- $\widehat{C}_{\mathcal{P}}$ measures the "non-commutative" rotation of the normal bundle along $\mathcal{F}_{\mathfrak{M}}$.

Non-triviality

Theorem: For each $n \geq 1$ and $q = 2n$, and homogeneous polynomial $C \in \mathcal{I}_q$ with $2\ell = \deg(C) \geq 2$, there exists a a generalized solenoid $\mathcal{S}$ over $B_0 = \mathbb{T}^n$ with tower $\mathcal{P}$ which satisfies the conditions in the embedding theorem above are satisfied for $r = 2$, so that $\mathcal{S} \subset \mathbb{T}^n \times \mathbb{D}^q$ is a C^2-embedding with the FPP with respect to $\mathcal{P}$, and the non-commutative self-linking invariant $\widehat{C}_{\mathcal{P}} \neq 0$.

That is, we can find smooth embeddings of generalized solenoids over $\mathbb{T}^n$ so that all of the higher-order linking numbers are non-trivial.

Application to classifying spaces

Let $B\Gamma_q^r$ denote the *Haefliger classifying space* of C^r codimension-q foliations.

Let $\mathfrak{M}$ be embedded as a minimal set of a C^r-foliation $\mathcal{F}$ of a foliated manifold M, with leaves of dimension n and codimension $q \geq 2$.

Let $B\mathcal{F}$ denote the classifying space of $\mathcal{F}$ on M, and $B\mathcal{F}_{\mathfrak{M}}$ its restriction to morphisms with source and target in $\mathfrak{M}$. Alternately, $B\mathcal{F}_{\mathfrak{M}}$ is the classifying space of the germ of $\mathcal{F}$ to an open neighborhood of $\mathfrak{M}$ in M.

Then there exists a well-defined "classifying map" $f_{\mathfrak{M}} \colon B\mathcal{F}_{\mathfrak{M}} \to B\Gamma_q^r$ defined by restricting the classifying map for $B\mathcal{F}$.

Theorem: For each $n \geq 1$, $q = 2n$, let $\mathcal{S}$ be a generalized solenoid which admits a C^2 embedding $\mathfrak{M} \subset \mathbb{T}^n \times \mathbb{D}^q$ as above. Then $f_{\mathfrak{M}} \colon B\mathcal{F}_{\mathfrak{M}} \to B\Gamma_q^r$ is non-trivial in cohomology for all degrees $2\ell - 1 > 2q$ for which the associated invariant $\widehat{C}_{\mathcal{P}} \neq 0$.

Ideas of proof

Bott and Heitsch observed that the cohomology of torsion flat bundles injects into the integral cohomology of the classifying space $B\Gamma_q$:

• [Bott & Heitsch] *A remark on the integral cohomology of* $B\Gamma_q$, Topology, 1972.

Cheeger and Simons observed that the Chern-Weil homomorphism for foliations and flat bundles can be used to calculate the torsion classes of the bundle:

• [Cheeger & Simons1985] *Differential characters and geometric invariants*, preprint, Stony Brook University, 1972.
• [Simons] *Characteristic forms and transgression: characters associated to a connection*, preprint, Stony Brook University, 1972.

In the last paragraph of this second preprint, Simons conjectures that the $\mathbb{R}$-valued differential characters associated to the normal bundle of foliations is non-trivial in all degrees above $2q$. We show this using embedded solenoids with the FPP.

Key steps

• Normal bundle to each approximation L_ℓ is flat with finite order p, so the Cheeger-Simons classes mod (p) inject into its classifying space.

• Classifying spaces for the leaves L_ℓ are embedded continuously into that for $\mathcal{F}$.

• Chern-Weil forms are defined on all of M, so vary continuously with the embedded cycles from the compact leaves.

• Limit of the compact cycles is a higher dimensional closed current on the semi-simplicial model for the classifying space $B\mathcal{F}_{\mathfrak{M}}$. Thus, the evaluation of the Chern-Weil forms on these asymptotic cycles is non-trivial.

• Finally, one notes that the classifying space of the germ of the embedding $\mathfrak{M} \subset M$ carries the inverse limit of the approximating tower $\mathcal{P}$ if the FPP condition holds, and that the Chern-Weil construction depends smoothly on data.

Conclusion

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Thank you for your attention.

slides available at
http://www.math.uic.edu/∼hurder