Entropy of the Kuperberg pseudogroup

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joint work with Ana Rechtman

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Counter-example to the Seifert Conjecture:

Theorem (K. Kuperberg, 1994) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

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Theorem (A. Katok, 1980) *Let M be a closed, orientable 3-manifold. Then an aperiodic flow ϕ_t on M has entropy zero.*

Theorem (Hurder & Rechtman, 2015) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞-family of non-vanishing vector fields $\vec{X}_t$ for $-\epsilon < t < \epsilon$ whose flows:*

- have entropy 0 for $t < 0$
- have no periodic orbits for $t = 0$
- have positive entropy for $t > 0$.

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Solution of Palis Conjecture for surface diffeomorphisms:

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Explicit family of constructions:

- S. Hurder and A. Rechtman, *Zippered laminations at the boundary of hyperbolicity*, preprint, 2015.

$\phi_t \colon M \to M$ a smooth flow. A *complete section* is a closed surface $T \subset M$ which is everywhere transverse to the flow.

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$\phi_t \colon M \to M$ a smooth flow. A section is a surface $T \subset M$ which is generically transverse to the flow.

Return map of flow induces a smooth pseudogroup $\mathcal{G}_\phi$ on T

Dynamics of $\phi_t \leftrightarrow$ dynamics of $\mathcal{G}_\phi$??

$\Sigma \subset M$ is minimal set for ϕ_t if:

- $\phi_t(\Sigma) = \Sigma$ for all t
- Σ is closed
- Σ is minimal with respect to these two properties.

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Suppose that section $T \cap \Sigma$ is contained in the interior of Σ

$\Longrightarrow$ Dynamics of $\phi_t \leftrightarrow$ dynamics of $\mathcal{G}_\phi$

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Theorem (Hurder & Rechtman, 2013) The unique minimal set for a generic Kuperberg flow is a *zippered lamination*, a stratified space with two strata:

- A 2-dimensional strata that has a laminated structure $\mathcal{F}$;
- A 1-dimensional strata that is transversally Cantor-like.
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$\Longrightarrow$ Dynamics of $\phi_t \leftrightarrow$ dynamics of $\mathcal{G}_\phi \leftrightarrow$ dynamics of $\mathcal{G}_{\mathcal{F}}$?

Theorem (Hurder & Rechtman, 2015) Entropy of Kuperberg flow vanishes if and only if geometric entropy of $\mathcal{G}_{\mathcal{F}}$ vanishes.

Strategy:

- Construct smooth variations of the Kuperberg flow
- Evaluate the entropy of these flows using pseudogroup entropy

Definition: A plug is a 3-manifold with boundary of the form $P = D \times [-1, 1]$ with D a compact surface with boundary. P is endowed with a non-vanishing vector field $\vec{X}$, such that:

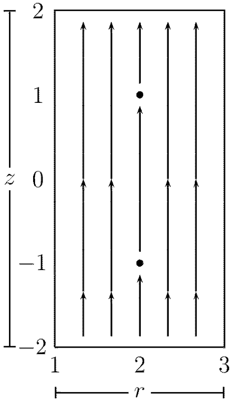
- $\vec{X}$ is vertical in a neighborhood of ∂P, that is $\vec{X} = \frac{d}{dz}$. Thus $\vec{X}$ is inward transverse along $D \times \{-1\}$ and outward transverse along $D \times \{1\}$, and parallel to the rest of ∂P.

- There is at least one point $p \in D \times \{-1\}$ whose positive orbit is trapped in P.

- If the orbit of $q \in D \times \{-1\}$ is not trapped then its orbit intersects $D \times \{1\}$ in the facing point.

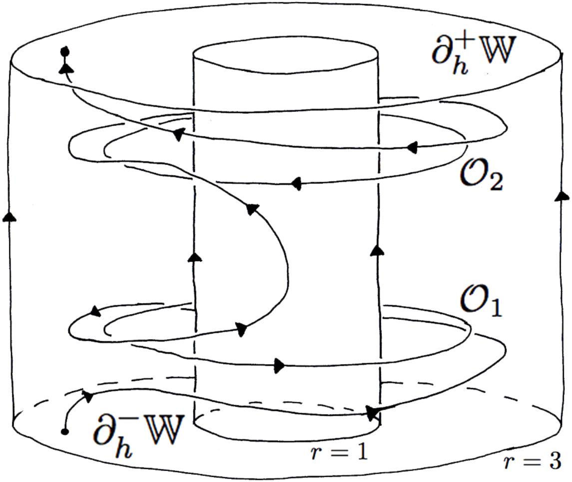
- There is an embedding of P into $\mathbb{R}^3$ preserving the vertical direction.

Construct a modified Wilson vector field $\vec{W}$ in a rectangle R

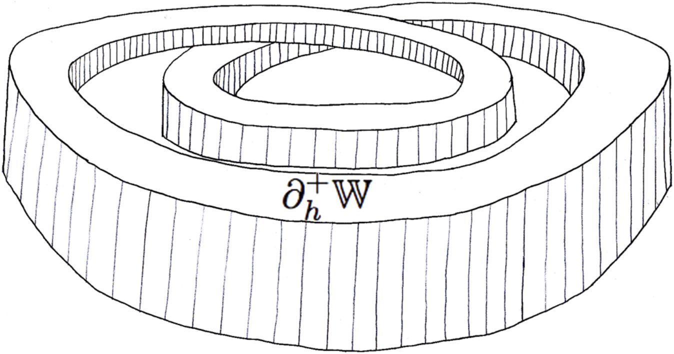


Modified Wilson Plug $\mathbb{W}$:

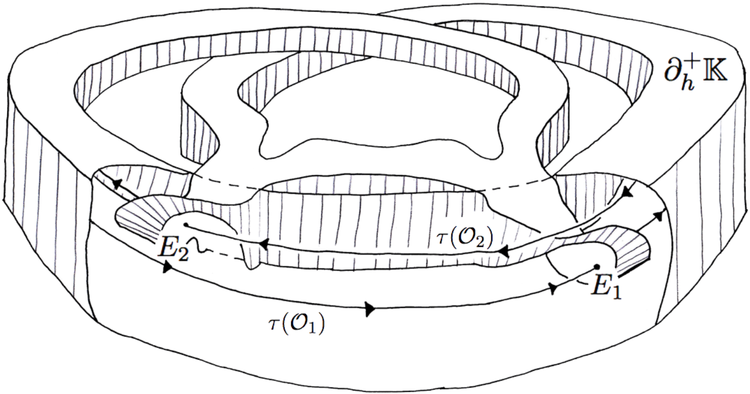
Consider the rectangle $R \times \mathbb{S}^1$ with the vector field $\vec{W} = \vec{W}_1 + f\,\frac{f}{d\theta}$
The function f is asymmetric in z.



Embed the Modified Wilson Plug in $\mathbb{R}^3$:

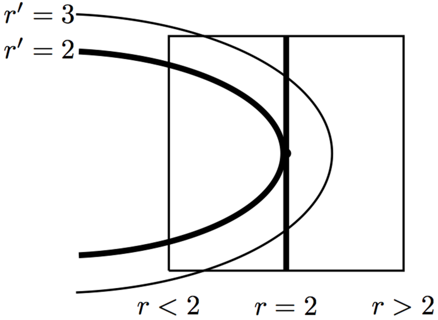


Grow horns and embed them to obtain Kuperberg Plug $\mathbb{K}$, matching the flow lines on the boundaries.

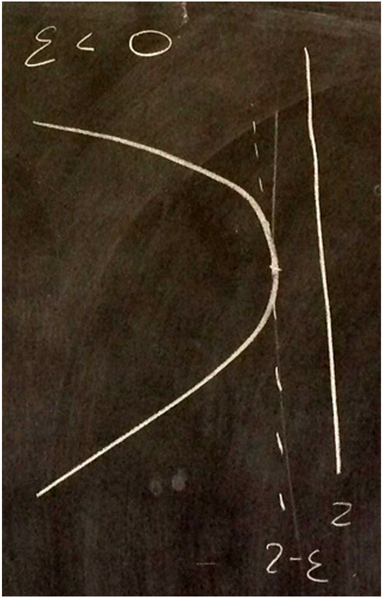


The subtlety: embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself. Or, vary this parameter by $-\epsilon < t < \epsilon$ to get $\vec{X}_t$ on $\mathbb{K}$

$\epsilon = 0$: The insertion map as it appears in the face E_1



$\epsilon < 0$: The insertion map as it appears in the face E_1



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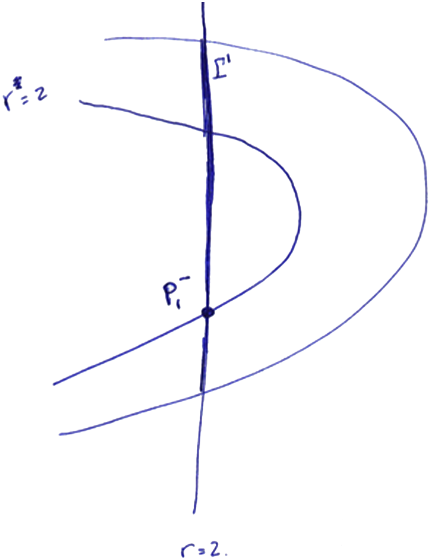
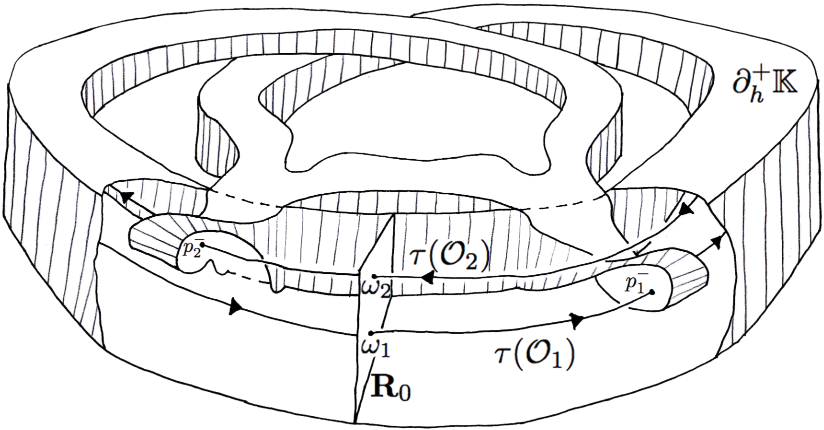
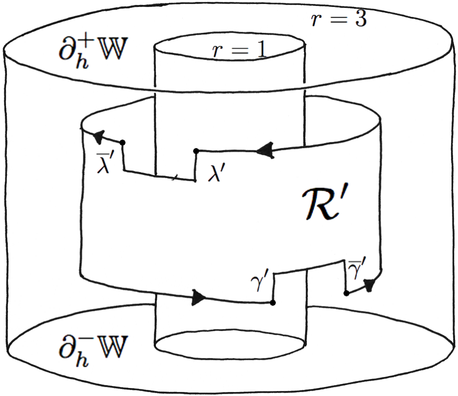


The section $\mathbf{R}_0 \subset \mathbb{K}$ used to define pseudogroups $\mathcal{G}_\phi$ and $\mathcal{G}_\mathcal{F}$.



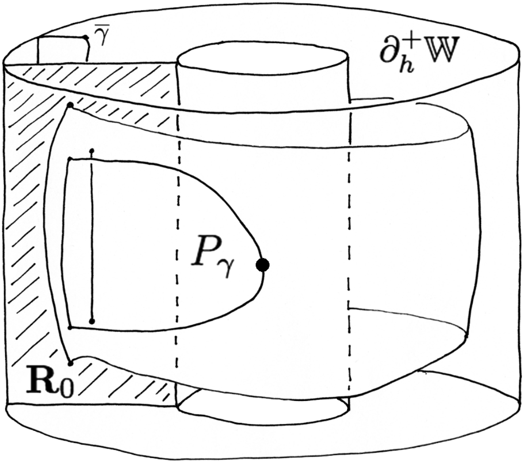
The flow $\vec{W}$ is tangent to $\mathbf{R}_0$ along the center plane $\{z = 0\}$.

The notched cylinder $\mathcal{R}'$ embedded in $\mathbb{W}$

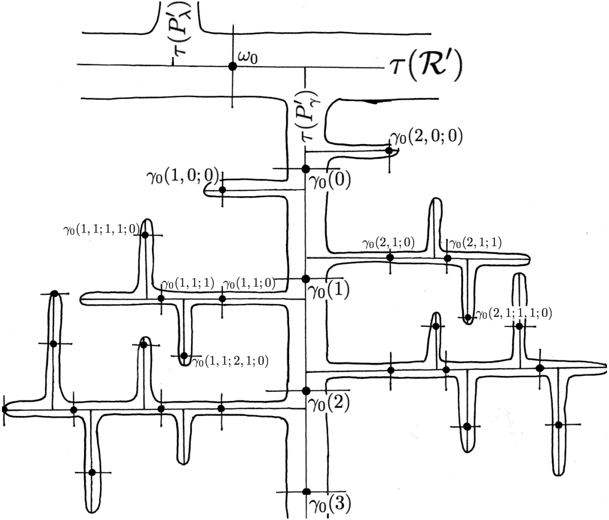


$\mathbb{K}$ is obtained from $\mathbb{W}$ by a quotient map, $\tau \colon \mathbb{W} \to \mathbb{K}$

The flow of the cylinder $\mathcal{R}'$ through one insertion is a simple propeller:



The flow of the cylinder $\mathcal{R}'$ through infinite time is an infinite branched tree $\mathfrak{M}_0$ - a leaf of a lamination:



The Kuperberg pseudogroup $\mathcal{G}_{\mathcal{F}}$ is generated by the holonomy of the lamination $\mathfrak{M}$ defined by the flow of the Reeb cylinder

$$\mathfrak{M}_0 \;\equiv\; \{\phi_t(\tau(\mathcal{R}')) \mid -\infty < t < \infty\}$$

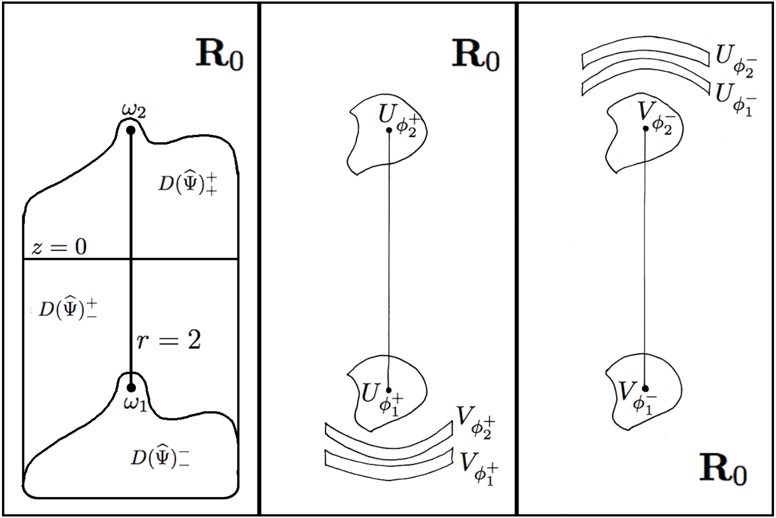
$$\mathfrak{M} \;\equiv\; \overline{\mathfrak{M}_0} \subset \mathbb{K}$$

$\mathfrak{M}$ is wildly wicked, though $\mathfrak{M}_0$ admits a level filtration

$$\mathfrak{M}_0^0 \subset \mathfrak{M}_0^1 \subset \mathfrak{M}_0^2 \subset \cdots$$

which matches the branching of the tree above.

Generators $\{Id, \phi_1^+, \phi_1^-, \phi_2^+, \phi_2^-, \psi\}$ of the pseudogroup $\mathcal{G}_{\mathcal{F}}$



The *word length* $\|g\| \leq m$ if $g \in \mathcal{G}_{\mathcal{F}}$ can be expressed as the composition of at most m generators.

Entropy for a C^1-pseudogroup action [Ghys, Langevin & Walczak, 1988] measures the "exponentiality" of the orbits.

Let $\epsilon > 0$ and $\ell > 0$, and d the metric on $\mathbf{R}_0$. A subset $\mathcal{E} \subset \mathbf{R}_0$ is said to be (d, ϵ, ℓ)-separated if for all $w, w' \in \mathcal{E}$ there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in \mathrm{Dom}(g)$, and $\|g\|_w \leq \ell$ so that $d(g(w), g(w')) \geq \epsilon$.

The "expansion growth function" is:

$$h(\mathcal{G}_{\mathcal{F}}, d, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathbf{R}_0 \text{ is } (d, \epsilon, \ell)\text{-separated}\}$$

The entropy of $\mathcal{G}_{\mathcal{F}}$ is the *asymptotic exponential growth type* of the expansion growth function:

$$h(\mathcal{G}_{\mathcal{F}}) = \lim_{\epsilon \to 0} \left\{ \limsup_{\ell \to \infty} \ \ln \left\{ h(\mathcal{G}_{\mathcal{F}}, d, \epsilon, \ell) \right\} / \ell \right\}$$

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Theorem (Hurder & Rechtman, 2015) Let $\vec{X}_t$ be the modified Kuperberg flow on $\mathbb{K}$. Then the "expansion growth function":

- for $-\epsilon < t < 0$ has polynomial growth, hence $h(\mathcal{G}_{\mathcal{F}}) = 0$
- for $t = 0$ has growth rate $\sim \ \exp(\sqrt{n})$, hence $h(\mathcal{G}_{\mathcal{F}}) = 0$
- for $0 < t < \epsilon$ has exponential growth, hence $h(\mathcal{G}_{\mathcal{F}}) > 0$.

Idea of proof:

- for $-\epsilon < t < 0$, the surface $\mathfrak{M}_0$ is finitely recursive
- for $t = 0$, the surface $\mathfrak{M}_0$ is partially recursive
- for $0 < t < \epsilon$, the surface $\mathfrak{M}_0$ is fully recursive.

Then use:

- Resulting growth estimates for number of words in $\mathcal{G}_{\mathcal{F}}$,
- estimate non-expansiveness of maps defined by the words in $\mathcal{G}_{\mathcal{F}}$.

References

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