Index Theory for Foliation Minimal Sets

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Introduction

We present an approach to the study of the topology, dynamics and spectral geometry of foliations, based on studying special types of "cycles", which combine "classical" ideas in the subject:

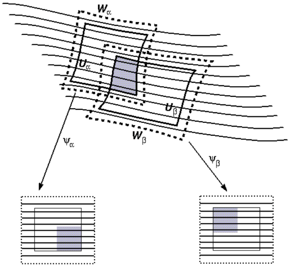
- Minimal sets for dynamical systems
- Classification of Cantor pseudogroup actions
- Spectral geometry of a foliation.

Problem: How are the spectra of leafwise elliptic geometric operators for $\mathcal{F}$ and the index theory for $\mathcal{F}$, related to the topology & Riemannian geometry of the leaves, and dynamical and topological properties of the foliation?

Foliation charts

Let M be a smooth manifold of dimension n.

Definition: M a smooth manifold is *foliated* if there is a covering of M by coordinate charts whose change of coordinate functions preserve leaves



Foliations

A foliation $\mathcal{F}$ of a compact manifold M is also . . .

• a "uniform partition" of M into submanifolds of dimension p

• a local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle for a "good covering". (Ehresmann, Haefliger)

• a dynamical system on M with multi-dimensional time.

• a groupoid $\Gamma_{\mathcal{F}} \to M$ with fibers complete manifolds, the holonomy covers of leaves.

Each point of view has advantages and disadvantages.

Foliation invariants

Each viewpoint yields its own classes of invariants:

• local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle
$\Longrightarrow$ Secondary classes; sheaf cohomology invariants.

• dynamical system on M with multi-dimensional time
$\Longrightarrow$ Geometric entropy, Lyapunov spectrum, invariant & harmonic measures.

• a "uniform partition" of M into submanifolds of dimension p
$\Longrightarrow$ spectrum of leafwise elliptic operators.

• groupoid $\Gamma_{\mathcal{F}} \to M$ with fibers the holonomy covers of leaves
$\Longrightarrow$ C^*-algebras $C^*_r(M, \mathcal{F})$ and its K-Theory invariants, von Neumann algebra $W^*(M, \mathcal{F})$ and its flow of weights.

Foliation C^*-algebras

$C_r^*(M/\mathcal{F})$ is a non-commutative C^*-algebra associated to fields of compact operators along the fibers of the *holonomy groupoid* $\Gamma_{\mathcal{F}} \to M$, with fibers the holonomy coverings of leaves of $\mathcal{F}$.

Problems: Use $C_r^*(M/\mathcal{F})$ to:

- Recover the "structure" of a foliation from its C^*-algebra.
- Determine geometric and dynamical properties of $\mathcal{F}$.

These questions are central to the study of non-commutative geometry of foliations. Partial answers have really only been given for special classes of foliations.

Closed invariant sets

Every dynamical system on a compact space has a collection of closed invariant sets.

$L \subset M$ a leaf of $\mathcal{F}$ in compact manifold, its closure $X = \overline{L}$ is closed and invariant. The minimal sets $\mathfrak{M} \subset \overline{L}$ have special significance.

Question: *Generalized Poincaré Recurrence principle:* dynamical properties of L and topological properties of $\mathfrak{M}$ are closely related.

- *How* is not at all clear, especially for codimension $q > 1$.
- Role of closed invariant sets is *fundamental*.
- Minimal sets are *foliated spaces*.

Foliated spaces

Definition: $\mathfrak{M}$ is an *n-dimensional foliated space* if:

$\mathfrak{M}$ is a compact metrizable space, and each $x \in \mathfrak{M}$ has an open neighborhood homeomorphic to $(-1,1)^n \times \mathfrak{T}_x$, where $\mathfrak{T}_x$ is a closed subset with interior of some Polish space $\mathfrak{X}$.

Natural setting for foliation index theorems:

Moore & Schochet, *Global Analysis on Foliated Spaces* [1988].

A foliated space $\mathfrak{M}$ can be a usual foliated manifold, or at the other extreme, a foliation of codimension zero.

Matchbox manifolds

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum: a compact, connected metrizable space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$;
- each $\mathfrak{T}_i$ is a clopen subset of a totally disconnected space $\mathfrak{X}$.

Then the arc-components of $\mathfrak{M}$ are locally Euclidean:

$$\mathfrak{T}_i \quad \text{are totally disconnected} \iff \mathfrak{M} \quad \text{is a matchbox manifold}$$

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, and the pseudogroup dynamics of the foliation $\mathcal{F}$ on the transverse fibers $\mathfrak{T}_i$ represents *intrinsic* fundamental groupoid.

The "matchbox manifold" concept is much more general than minimal sets for foliations: they also appear in study of tiling spaces, subshifts of finite type, graph constructions, generalized solenoids, pseudogroup actions on totally disconnected spaces, . . .

Examples: Minimal $\mathbb{Z}^n$-actions on Cantor set K.

- Adding machines (minimal equicontinuous systems)
- Toeplitz subshifts over $\mathbb{Z}^n$
- Minimal subshifts over $\mathbb{Z}^n$

More examples

Replace $\mathbb{Z}^n$ by an finitely generated group Γ, the torus $\mathbb{T}^n$ by a compact manifold B with $\pi_1(B, b_0) \cong \Gamma$, choose a transitive action of Γ on a Cantor set, and form the suspension foliation.

Leaves are coverings of the base space B.

This gives a large collection of examples, but these are just a sampling of the variety of matchbox manifolds.

Vietoris solenoids

The classical solenoid is the *Vietoris solenoids*, defined by a tower of covering maps of $\mathbb{S}^1$. For $\ell \geq 0$, given orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \mathbb{S}^1 \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

Then the p_ℓ are the *bonding maps* of degree $p_\ell > 1$ for the solenoid

$$\mathcal{S} = \lim_{\leftarrow} \; \{p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \; \subset \; \prod_{\ell=0}^{\infty} \mathbb{S}^1$$

Proposition: $\mathcal{S}$ is a continuum with an equicontinuous flow $\mathcal{F}$, so is a 1-dimensional matchbox manifold.

Weak solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving proper covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \lim_{\leftarrow} \{p_\ell \colon B_\ell \to B_{\ell-1}\} \subset \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: $\mathcal{S}$ has natural structure of a matchbox manifold, with every leaf dense. The dynamics of $\mathcal{F}$ are *equicontinuous*.

- Topologies of leaves for these examples are not well-understood.

McCord solenoids

Basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

Definition: A weak solenoid $\mathcal{S}$ is a *McCord solenoid*, if for some fixed $\ell_0 \geq 0$, then for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}.

Classifying weak solenoids

A weak solenoid is "determined" by the base manifold B_0 and the tower equivalence of the inverse chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

Topology of leaves in McCord solenoids are determined by algebraic properties of the tower.

Theorem: [Pontrjagin 1934; Baer 1937] $G_0 \cong \mathbb{Z}$, homeomorphism types of McCord solenoids are uncountable, but classifiable.

Theorem: [Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^n$ with $n \geq 2$, the homeomorphism types of McCord solenoids are *not* classifiable. (in the sense of Descriptive Set Theory)

The number of such is not just huge, but indescribably large.

Problem: How can these be distinguished, at least in part, using spectral invariants?

Pseudogroups

Dynamics of $\mathfrak{M}$ defined by a *pseudogroup* action on a Cantor set.

Covering of $\mathfrak{M}$ by foliation charts $\Longrightarrow$ transversal $\mathcal{T} \subset \mathfrak{M}$ for $\mathcal{F}$

Holonomy of $\mathcal{F}$ on $\mathcal{T} \Longrightarrow$ pseudogroup $\mathcal{G}_{\mathcal{F}}$ generated by maps on clopen sets:

- compact clopen set $\mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite generating set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$;
- $g_i \colon D(g_i) \to R(g_i)$ is restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$, $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Topological dynamics

Dynamical properties of $\mathcal{F}$ formulated in terms of $\mathcal{G}_{\mathcal{F}}$; e.g.,

$\mathcal{F}$ has no leafwise holonomy if for $g \in \mathcal{G}_{\mathcal{F}}$, $x \in Dom(g)$, $g(x) = x$ implies $g|V = Id$ for some open neighborhood $x \in V \subset \mathcal{T}$.

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \implies d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

Equicontinuous matchbox manifolds

Analogs of Riemannian foliations. Can they be classified?

"Molino Theorem" for matchbox manifolds:

Theorem: [Clark & H 2011] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathcal{F}$ is minimal, and

• $\mathfrak{M}$ homeomorphic to a weak solenoid, so is homeomorphic to the suspension of an minimal equicontinuous action of a countable group on a Cantor space $\mathbb{K}$.

• $\mathfrak{M}$ homogeneous $\Longrightarrow$ homeomorphic to a McCord solenoid

We know "what they are", but cannot classify - just too many.

Expansive matchbox manifolds

Definition: $\mathfrak{M}$ is an *expansive matchbox manifold* if it admits some covering by foliation charts, and there exists $\epsilon > 0$, so that for all $w \neq w' \in \mathcal{T}_i$ for some $1 \leq i \leq k$ with $d_{\mathcal{T}}(w, w') < \epsilon$, then there exists a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{T}$ with $w, w' \in \mathrm{Dom}(h_{\tau_{w,z}})$ such that $d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) \geq \epsilon$.

Example: Denjoy minimal sets are expansive.

Example: Let Δ be a *quasi-periodic tiling* of $\mathbb{R}^n$, which is *repetitive*, *aperiodic*, and has *finite local complexity*, then the "hull closure" Ω_Δ of the translates of Δ by the action of $\mathbb{R}^n$ defines an expansive matchbox manifold $\mathfrak{M}$.

Dynamics dichotomy

Theorem: [Auslander–Glasner–Weiss, 2007] Let Γ be a finitely generated group acting minimally on a Cantor set $\mathfrak{X}$. If the action is distal, then it must be equicontinuous.

This result suggests:

Conjecture: Let $\mathfrak{M}$ be a minimal matchbox manifold. Then either the holonomy action of $\mathcal{G}_{\mathcal{F}}^{\mathfrak{X}}$ on $\mathfrak{X}$ equicontinuous, or it is expansive.

Generalized solenoids

A generalized solenoid is defined by a tower of *branched manifolds*

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The *bonding maps* p_ℓ are assumed to be locally smooth cellular maps, but not necessarily covering maps. Set:

$$\mathcal{S} = \varprojlim \, \{p_\ell \colon B_\ell \to B_{\ell-1}\} \ \subset \ \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: If the local degrees of the maps p_ℓ tend to ∞, then the inverse limit $\mathcal{S}$ has natural structure of a matchbox manifold.

These are more general than the *Williams solenoids*, but same idea.

Theorem: (Anderson & Putnam [1991], Sadun [2003]) A *quasi-periodic tiling* Δ of $\mathbb{R}^n$, which is *repetitive*, *aperiodic*, and has *finite local complexity*, then the "hull closure" Ω_Δ of the translates of Δ by the action of $\mathbb{R}^n$ is homeomorphic to a generalized solenoid.

Benedetti & Gambaudo [2003] extended this to the case of tilings on a connected Lie group G, in place of the classical case $\mathbb{R}^n$.

Theorem: [Clark, H, Lukina 2012] Let $\mathfrak{M}$ be a minimal matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a generalized solenoid.

For a minimal set in codimension-one foliation, this coincides with what we know about the shape of these exceptional minimal sets.

Theorem: Let $\mathfrak{M}$ be an exceptional minimal set for a smooth foliation $\mathcal{F}$ of compact manifold M. Then $\mathfrak{M}$ admits a presentation as an inverse tower of compact branched manifolds B_ℓ. Moreover, given $\epsilon > 0$ there is a compact, orientable base B_{ℓ_0} and a tower of proper orientation-preserving "covering maps"

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} \cdots \xrightarrow{p_{\ell_0+2}} B_{\ell_0+1} \xrightarrow{p_{\ell_0+1}} B_{\ell_0}$$

where each B_ℓ embeds in M as an "approximate leaf" for $\mathcal{F}$, so that the leaves of $\mathcal{F}|\mathfrak{M}$ are "coverings" of B_ℓ that lie in an ϵ-tube around B_ℓ.

Problem: What does this structure theory imply for foliation spectral geometry?

M is a closed Riemannian manifold.

$\mathcal{F}_M$ is an n-dimensional foliation on M.

$\mathcal{E} \to M$ an Hermitian bundle, for which there is a first-order geometric operator along the leaves of $\mathcal{F}$,

$$\mathcal{D}\colon C^\infty(\mathcal{F}, \mathcal{E}) \to C^\infty(\mathcal{F}, \mathcal{E})$$

That is, for each leaf L of $\mathcal{F}_M$ the restriction

$$\mathcal{D}_L\colon C_c^\infty(L, \mathcal{E}_L) \to C_c^\infty(L, \mathcal{E}_L)$$

is elliptic and essentially self-adjoint.

For each leaf L of $\mathcal{F}$, the restricted operator

$$\mathcal{D}_L \colon C_c^\infty(L, \mathcal{E}_L) \to C_c^\infty(L, \mathcal{E}_L)$$

is elliptic and essentially self-adjoint, so we can define its spectrum, and the decomposition into pure and continuous parts:

$$\sigma(\mathcal{D}_L) = \sigma_a(\mathcal{D}_L) \cup \sigma_c(\mathcal{D}_L) \subset \mathbb{R}$$

In general, very little is known about this. For related discussions:

Topology of covers and the spectral theory of geometric operators, in *Index theory and operator algebras (Boulder 1991)*, Contemp. Math. Vol. 148, Amer. Math. Soc., 1993.

Gap Labeling & K-Theory

Spectral flow approach to spectrum via odd index theory for foliations, as in the solutions of Bellisard's "Gap Labeling Conjecture" in the study of quasi-crystals on $\mathbb{R}^n$:

- Kaminker & Putnam [2003]
- Benameur & Oyono-Oyono [2003]
- Bellissard, Benedetti & Gambaudo [2006]

Idea: Study the spectral problem on exceptional minimal sets $\mathfrak{M}$: consider spectral problems as "gap labeling" questions involving spectral flow of geometric operators along leaves of $\mathfrak{M}$.

KK-Index

$C^*(M/\mathcal{F}_M)$ is the *reduced* C^*-algebra associated to $\mathcal{F}_M$.

Theorem: [Connes-Fack 1984] A first-order geometric operator $\mathcal{D}$ along the leaves of $\mathcal{F}$ defines a generalized index class

$$[\mathcal{D}] \in KK(M, M/\mathcal{F}) \cong KK(C(M), C^*(M/\mathcal{F}))$$

Given a K-Theory class $[E] \in K(M)$ we obtain the "index class"

$$[E] \cap [\mathcal{D}] = \mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E) \in K^*(C^*(M/\mathcal{F}))$$

Transverse measures and traces

Let μ be a holonomy-invariant transverse measure for the foliation $\mathcal{F}$. That is, for a Borel subset $E \subset M$ which intersects each leaf L at most countable many times, then $\mu(E) \in \mathbb{R}$ is invariant under the holonomy translations of $\mathcal{F}$.

Associated to μ is a trace $Tr_\mu \colon C^*(M/\mathcal{F}) \to \mathbb{R}$.

Proposition: The μ-dimension function $Tr_\mu \colon K^0(C^*(M/\mathcal{F})) \to \mathbb{R}$ is well-defined, and measures the differences of the *von Neumann dimensions* of the projections in $C^*(M/\mathcal{F})$ defining a class.

Transverse measures and foliated spaces

The support $\mathcal{Z}(\mu) \subset M$ is the smallest closed saturated subset for which μ has "full measure".

Lemma: $\mathcal{Z}(\mu)$ is a foliated space.

So, we can restrict $\mathcal{D}$ to the leaves of $\mathcal{F}|\mathcal{Z}(\mu)$ and study the meaning of the index pairing on this space.

Motivates the approach in the book by Moore & Schochet.

Problem: What is the analytic meaning of the restricted index? How does it depend on the dynamics of $\mathcal{F}|\mathcal{Z}(\mu)$?

Generic invariant sets

Let $\mathcal{Z} \subset M$ be a closed saturated subset, with foliation $\mathcal{F}$.

Definition: $\mathcal{Z}$ is *generic* if each leaf $L \subset \mathfrak{M}$ without holonomy for $\mathcal{F}$, is also without holonomy as a leaf for $\mathcal{F}$ on M.

If the leaves of $\mathcal{F}$ are all simply connected, then $\mathcal{Z}$ is generic.

Proposition: [Fack & Skandalis 1982] If $\mathcal{Z} \subset M$ is a generic closed invariant set, then there is a well-defined restriction map

$$\iota_{M,\mathcal{Z}} \colon C^*(M/\mathcal{F}) \to C^*(\mathcal{Z}/\mathcal{F})$$

Consequently we obtain the restricted index class:

$$\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E|\mathcal{Z}) = [E] \cap [\mathcal{D}] \cup [\iota_{M,\mathcal{Z}}] \in K^*(C^*(\mathcal{Z}/\mathcal{F}))$$

Corollary: Let $\mathcal{Z} \subset M$ be a generic closed invariant set containing the support of a holonomy invariant transverse measure μ. Then there is a well defined measured-index functional

$$\mathrm{Ind}_{\mu} \equiv Tr_{\mu} \circ \mathrm{Ind}_{\mathcal{F}} \circ \iota^{*}_{M,\mathcal{Z}} \colon K_{*}(M) \to \mathbb{R}$$

which factors through $K^{*}(C^{*}_{r}(\mathcal{Z}/\mathcal{F})$.

We can then consider $\mathcal{Z} \subset M$ as the image of a foliated space $\mathfrak{M}$ and study our problems in this context.

Assumptions:

- $\mathfrak{M}$ is a leafwise orientable matchbox manifold;
- The leaves of $\mathcal{F}$ are simply connected.
- There is a holonomy invariant measure μ supported on $\mathfrak{M}$.

Geometric interpretation of even index

Suppose that the leaves of $\mathcal{F}$ have even dimension.

Theorem: Let $[E] \in K^0(M)$. If $\mathcal{F}|\mathcal{Z}$ has equicontinuous dynamics, then there is a constant $\lambda_0 > 0$ so that the measured index

$$Tr_\mu(\mathrm{Ind}_\mathcal{F}(\mathcal{D} \otimes E|\mathcal{Z})) = \lim_{\ell \to \infty} \frac{\lambda_0}{d(\ell, \ell_0)} \cdot \mathrm{Ind}(\mathcal{D}_\ell \otimes E|B_\ell)$$

where

- λ_0 depends only on μ and the choice of B_{ℓ_0}
- $d(\ell, \ell_0)$ is the covering degree of $q_\ell \colon B_\ell \longrightarrow B_{\ell_0}$
- $E|B_\ell$ is restriction of $E \to M$ to an embedding of $B_\ell \subset M$
- $\mathcal{D}_\ell$ is a geometric operator on B_ℓ which *approximates* the leafwise operator $\mathcal{D}$.

This result should be compared with the geometric interpretation given to the index for measured laminations by surfaces in:

The ∂-operator, *Appendix A*, *Global analysis on foliated spaces*, by C. C. Moore and C. Schochet, 1988.

The above result gives an "extension" of the conclusions there.

The assumption that the dynamics of the lamination is *equicontinuous* implies the existence of approximating cycles $B_\ell \subset M$, and these cycles "carry" the measured index.

Geometric interpretation of odd dimension

Suppose that the leaves of $\mathcal{F}$ have odd dimension.

Theorem: Let $\varphi \colon M \to U(N)$ be a leafwise smooth function with values in the unitary group, for some $N > 0$. If $\mathcal{F}|\mathcal{Z}$ has equicontinuous dynamics, then there is a constant $\lambda_0 > 0$ so that the measured index

$$\eta_\mu(\mathcal{D}, \varphi) = Tr_\mu(\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes \varphi|\mathcal{Z})) = \lim_{\ell \to \infty} \frac{\lambda_0}{d(\ell, \ell_0)} \cdot \eta(\mathcal{D}_\ell, \varphi)$$

where $\eta_\mu(\mathcal{D}, \varphi)$ is the leafwise η-invariant, and $\eta(\mathcal{D}_\ell, \varphi)$ denotes the relative η-invariant for $\mathcal{D}_\ell$ coupled to the twisted flat bundle over B_ℓ defined by the restricted unitary bundle $\varphi|B_\ell$.

This result should be compared with the geometric interpretation given to the odd index for flat bundles in the papers:

Toeplitz operators and the eta invariant: the case of S^1, (with Ronald G. Douglas and Jerome Kaminker), in *Index theory of elliptic operators, foliations, and operator algebras*, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988.

Eta invariants and the odd index theorem for coverings, in *Geometric and topological invariants of elliptic operators*, Contemp. Math., 105, Amer. Math. Soc., Providence, RI, 1990.

Both of these papers implicitly assumed that the dynamics of the foliations being considered are equicontinuous.

For all exceptional minimal sets $\mathcal{Z} \subset M$ there is a tower of approximations of $\mathcal{Z}$ by "smooth" branched submanifolds.

Problem: Suppose the dynamics of $\mathcal{F}|\mathcal{Z}$ is not equicontinuous, then what does index theory on branched manifolds mean?

Do the above results have counterparts in this case?

Thank you for your time and attention.