Smooth flows with fractional entropy dimension

Steve Hurder

MCA – Montreal, July 27, 2017

University of Illinois at Chicago
www.math.uic.edu/∼hurder

Theorem (K. Kuperberg, 1994) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

- K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.

Theorem (K. Kuperberg, 1994) *Let M be a closed, orientable 3-manifold. Then M admits a C^∞ non-vanishing vector field whose flow ϕ_t has no periodic orbits.*

- K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.

There are many choices in the construction of Kuperberg plugs:

> *Ghys: Par ailleurs, on peut construire beaucoup de pièges de Kuperberg et il n'est pas clair qu'ils aient le même dynamique.*

- É. Ghys, *Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)*, Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, **Astérisque**, 227: 283–307, 1995.

Problem: Investigate the invariants of “Kuperberg flows”:

- dynamical invariants of the smooth flow in plug $\mathbb{W}$
- topological invariants of unique minimal set Σ
- relations with their smooth deformations.

Problem: Investigate the invariants of "Kuperberg flows":

- dynamical invariants of the smooth flow in plug $\mathbb{W}$
- topological invariants of unique minimal set Σ
- relations with their smooth deformations.

Theorem (Katok, 1980) *Let M be a closed, orientable 3-manifold. A smooth aperiodic flow ϕ_t on M has entropy zero.*

Theorem (H & Rechtman, 2016) *The minimal set Σ of a generic Kuperberg flow is a 2-dimensional "zippered lamination", which has unstable shape.*

Theorem (Ingebretson, 2017) *The Hausdorff dimension of the minimal set Σ for a generic Kuperberg flow has $2 < HD(\Sigma) < 3$.*

Let $\varphi_t \colon M \to M$ be a smooth non-vanishing flow on a compact Riemannian manifold. For $\epsilon, T > 0$, two points $p, q \in M$ are said to be (φ_t, T, ϵ)-*separated* if

$$d_M(\varphi_t(p), \varphi_t(q)) > \epsilon \quad \text{for some} \quad -T \leq t \leq T \ .$$

A set $E \subset M$ is (φ_t, T, ϵ)-*separated* if all pairs of distinct points in E are (φ_t, T, ϵ)-separated. Let $s(\varphi_t, T, \epsilon)$ be the maximal cardinality of a (φ_t, T, ϵ)-separated set in X.

The topological entropy of the flow φ_t is then defined by

$$h_{top}(\varphi_t) = \frac{1}{2} \cdot \lim_{\epsilon \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{T} \log(s(\varphi_t, T, \epsilon)) \right\} \ ,$$

which is independent of the choice of metric d_M.

For a flow with zero entropy, de Carvalho, and independently Katok and Thouvenot, introduced the notion of *slow entropy* as a measure of the complexity of the flow. The slow entropy measures the subexponential growth of the ϵ-separated points.

Definition. For $0 < \alpha < 1$, the α-slow entropy of φ_t is given by

$$h^{\alpha}_{top}(\varphi_t) = \frac{1}{2} \cdot \lim_{\epsilon \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{T^{\alpha}} \log\{s(\varphi_t, T, \epsilon)\} \right\} .$$

For a flow with zero entropy, de Carvalho, and independently Katok and Thouvenot, introduced the notion of *slow entropy* as a measure of the complexity of the flow. The slow entropy measures the subexponential growth of the ϵ-separated points.

Definition. For $0 < \alpha < 1$, the α-slow entropy of φ_t is given by

$$h^{\alpha}_{top}(\varphi_t) = \frac{1}{2} \cdot \lim_{\epsilon \to 0} \left\{ \limsup_{T \to \infty} \frac{1}{T^{\alpha}} \log\{s(\varphi_t, T, \epsilon)\} \right\} .$$

Kyewon Park and her coauthors introduced the notion of the *entropy dimension* of a flow φ_t:

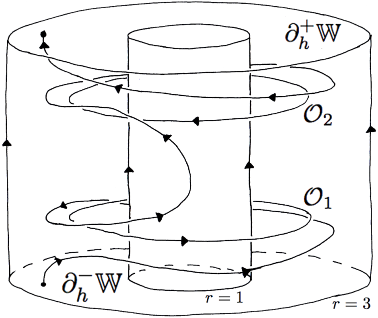
$$\mathrm{Dim}_h(\varphi_t) = \inf_{\alpha > 0} \left\{ h^{\alpha}_{top}(\varphi_t) \right\} = 0 .$$

- Are there non-trivial entropy-like invariants for Kuperberg flows?

$\mathbb{W} = [-2, 2] \times [1, 3] \times \mathbb{S}^1$ with non-vanishing vector field

$$\vec{W} = g\frac{f}{dz} + f\frac{f}{d\theta}$$

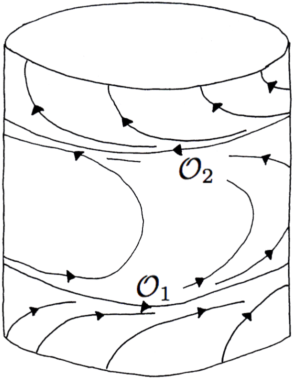
- f is asymmetric in the vertical coordinate z about $z = 0$
- $g \geq 0$ is constant in the $\mathbb{S}^1$ factor, and vanishes only along the circles $\mathcal{O}_i = \{(-1)^i\} \times \{2\} \times \mathbb{S}^1$



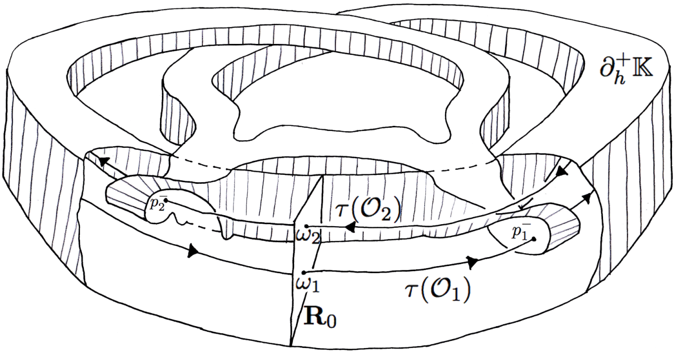
By symmetry on $g \geq 0$, it must vanish to an even order along $\mathcal{O}_i$.

In the generic case, g vanishes to second order.

Consider the case where g vanishes to order $2n$ for $n > 1$. As n increases, the speed of approach to the orbits $\mathcal{O}_i$ slows down.



Self-insert the Wilson plug with a twist and a bend, matching the flow lines on the boundaries, to obtain Kuperberg Plug $\mathbb{K}$



Embed so that the Reeb cylinder $\{r = 2\}$ is tangent to itself.

The degree of tangency influences the dynamics.

Define the closed subsets of $\mathbb{W} = [1,3] \times \mathbb{S}^1 \times [-2,2] \cong \mathbf{R} \times \mathbb{S}^1$

$\mathcal{D}_i = \sigma_i(D_i)$ for $i = 1, 2$ are solid 3-disks embedded in $\mathbb{W}$.

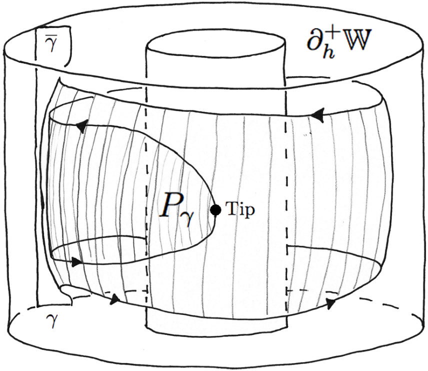
$$\mathbb{W}' \equiv \mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\} \quad , \quad \widehat{\mathbb{W}} \equiv \overline{\mathbb{W} - \{\mathcal{D}_1 \cup \mathcal{D}_2\}} \, .$$

$$\begin{aligned} \mathcal{C} &\equiv \{r = 2\} \quad [\textit{Full Cylinder}] \\ \mathcal{R} &\equiv \{(2, \theta, z) \mid -1 \leq z \leq 1\} \quad [\textit{Reeb Cylinder}] \\ \mathcal{R}' &\equiv \mathcal{R} \cap \widehat{\mathbb{W}} \quad [\textit{Notched Reeb Cylinder}] \\ \mathcal{O}_i &\equiv \{(2, \theta, (-1)^i)\}, i = 1, 2 \quad [\textit{Periodic Orbits}] \end{aligned}$$

Consider the flow of the image $\tau(\mathcal{R}') \subset \mathbb{K}$

$$\mathfrak{M}_0 \equiv \{\Phi_t(\tau(\mathcal{R}')) \mid -\infty < t < \infty\}$$

The surface $\mathfrak{M}_0$ is union of embedded "tongues" in $\mathbb{K}$, where each tongue wraps around the Reeb cylinder $\tau(\mathcal{R}')$.



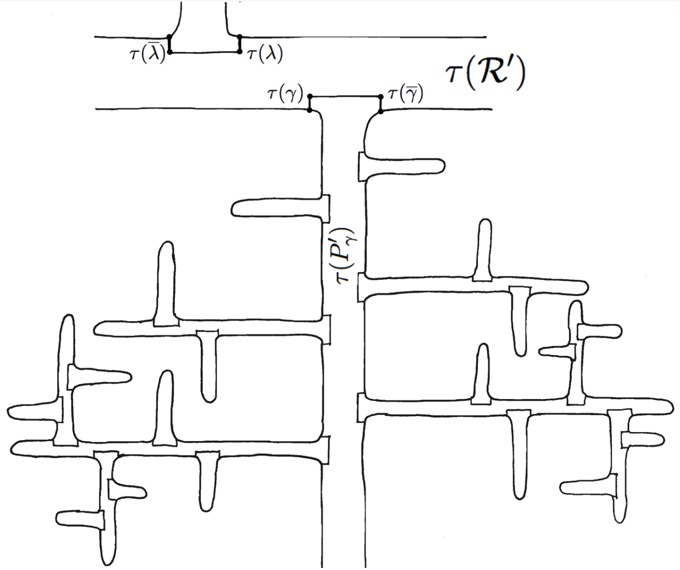
$\mathfrak{M}_0$ is an infinite union of tongues at increasing levels, corresponding to the level filtration

$$\mathfrak{M}_0^0 \subset \mathfrak{M}_0^1 \subset \mathfrak{M}_0^2 \subset \cdots$$

The closure $\mathfrak{M} \equiv \overline{\mathfrak{M}_0} \subset \mathbb{K}$ is a lamination with boundary.

The topology of $\mathfrak{M}$ is highly complex.

$\mathfrak{M}_0$ is a "fat tree" whose leaves at higher levels are recurrent on themselves, corresponding to the branching of the tree below.


$\tau(\bar{\lambda})$
$\tau(\lambda)$
$\tau(\mathcal{R}')$
$\tau(\gamma)$
$\tau(\bar{\gamma})$
$\tau(P'_\gamma)$

Choose a Riemannian metric on the plug $\mathbb{K}$.

Then $\mathfrak{M}_0 \subset \mathbb{K}$ inherits a Riemannian metric.

Let $d_{\mathfrak{M}}$ denote the associated path-distance function on $\mathfrak{M}_0$.

Fix the basepoint $\omega_0 = (2, \pi, 0) \in \tau(\mathcal{R}')$ and let

$$B_{\omega_0}(s) = \{x \in \mathfrak{M}_0 \mid d_{\mathfrak{M}}(\omega_0, x) \leq s\}$$

be the closed ball of radius s about the basepoint ω_0.

Let $A(X)$ denote the Riemannian area of a Borel subset $X \subset \mathfrak{M}_0$.

Then $\mathrm{Gr}(\mathfrak{M}_0, s) = A(B_{\omega_0}(s))$ is the *growth function* of $\mathfrak{M}_0$.

Given functions $f_1, f_2 \colon [0, \infty) \to [0, \infty)$, we say that $f_1 \lesssim f_2$ if there exists constants $A, B, C > 0$ such that for all $s \geq 0$, we have that $f_2(s) \ \leq \ A \cdot f_1(B \cdot s) + C$.

Say that $f_1 \sim f_2$ if both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ hold.

$f_1 \sim f_2$ defines an equivalence relation on functions, which is used to define their *growth type*.

Theorem (H & Ingebretson, 2017) There exists Kuperberg flows such that the growth type $\mathrm{Gr}(\mathfrak{M}_0, s)$ satisfies

$$\mathrm{Gr}(\mathfrak{M}_0, s) \sim \exp(s^{\alpha})$$

for $\alpha > 0$ arbitrarily small.

We give two applications of this construction.

The Kuperberg pseudogroup $\mathcal{G}_{\mathcal{F}}$ is generated by the holonomy of the lamination $\mathfrak{M}$ for the section $\mathbf{R}_0$.

The "expansion growth function" is:

$$h(\mathcal{G}_{\mathcal{F}}, d, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathbf{R}_0 \text{ is } (d, \epsilon, \ell)\text{-separated}\}$$

The complexity of $\mathcal{G}_{\mathcal{F}}$ is the *growth type* of $\ell \mapsto h(\mathcal{G}_{\mathcal{F}}, d, \epsilon, \ell)$

Theorem (H & Ingebretson, 2017) There exists Kuperberg flows such that for $\epsilon > 0$ sufficiently small, the growth type satisfies $h(\mathcal{G}_{\mathcal{F}}, d, \epsilon, \ell) \sim \exp(\ell^{\alpha})$, for $\alpha > 0$ arbitrarily small.

These examples have non-trivial *lamination* slow entropy.

The relation between the lamination slow entropy and the flow slow entropy is complicated.

The open sets $U_\ell = \{x \in \mathbb{K} \mid d_{\mathbb{K}}(x, \Sigma) < \epsilon_\ell\}$ where we have $0 < \epsilon_{\ell+1} < \epsilon_\ell$ for all $\ell \geq 1$, and $\lim_{\ell \to \infty} \epsilon_\ell = 0$, give a shape approximation to Σ.

For $\alpha > 0$, an α-pseudo-orbit for the Kuperberg flow φ_t determines a path in U_ℓ if $\alpha < \epsilon_\ell$.

Theorem (Misiurewicz, 1984) $h_{top}(\varphi_t) = h_\psi(\varphi_t)$ where $h_\psi(\varphi_t)$ denotes the entropy of φ_t calculated using pseudo-orbits.

Theorem (Barge & Swanson, 1990) $h_{top}(\varphi_t) = H_\psi(\varphi_t)$ where $H_\psi(\varphi_t)$ denotes the growth rate of separated periodic pseudo-orbits for φ_t.

Conjecture: The expansion growth function for φ_t defined using pseudo-orbits has the same growth type as for $\mathfrak{M}_0$.

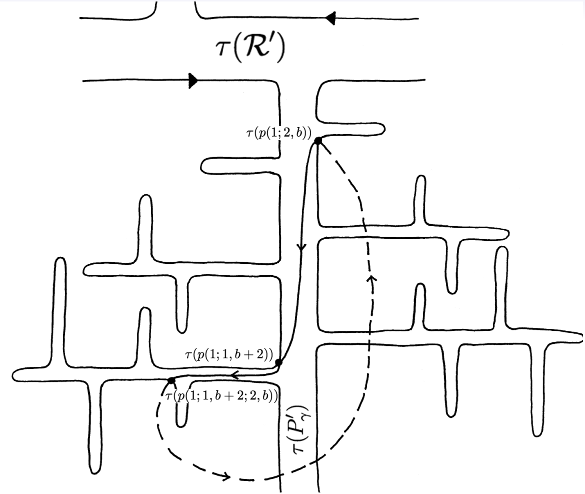
We also state an additional shape property for the minimal set of a generic Kuperberg flow.

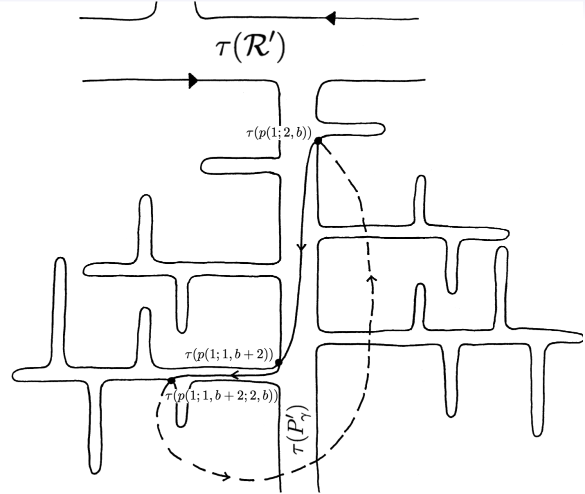
Theorem (H & Rechtman, 2016) Let Σ be the minimal set for a generic Kuperberg flow. Then the Mittag-Leffler condition for homology groups is satisfied. That is, given a shape approximation $\mathfrak{U} = \{U_\ell\}$ for Σ, then for any $\ell \geq 1$ there exists $p > \ell$ such that for any $q \geq p$

$$Image\{H_1(U_p; \mathbb{Z}) \to H_1(U_\ell; \mathbb{Z})\} = Image\{H_1(U_q; \mathbb{Z}) \to H_1(U_\ell; \mathbb{Z})\}.$$

A shape 1-cycle is an "almost closed" path with endpoints sufficiently close (see picture below.)

Problem: How are the shape 1-cycles related to the periodic pseudo-orbits for φ_t?


$\tau(\mathcal{R}')$
$\tau(p(1;2,b))$
$\tau(p(1;1,b+2))$
$\tau(p(1;1,b+2;2,b))$
$\tau(P'_\gamma)$



Thank you for your attention!

References

K. Kuperberg, *A smooth counterexample to the Seifert conjecture*, **Ann. of Math. (2)**, 140:723–732, 1994.

S. Hurder and A. Rechtman, *The dynamics of generic Kuperberg flows*, **Astérisque**, Vol. 377, 2016.

S. Hurder and A. Rechtman, *Aperiodicity at the boundary of chaos* , **Ergodic Theory Dynamical Systems**, to appear, 2017.

S. Hurder and A. Rechtman, *Perspectives on Kuperberg flows*, **Topology Proceedings**, to appear, 2017.

D. Ingebretson, *Hausdorff dimension of generic Kuperberg minimal sets*, **Thesis**, University of Illinois at Chicago, August 2017.

S. Hurder and D. Ingebretson, *Smooth flows with fractional entropy dimension*, in preparation, 2017.

S. Hurder and A. Rechtman, *The dynamics of Derived from Kuperberg flows*, in preparation, 2017.