Symmetries of laminations

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Arithmetic and Analysis

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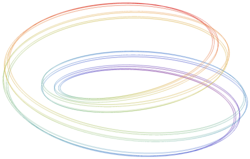
Christopher Deninger's talk at the International Congress of Mathematicians in Berlin, 1998 highlighted some deep analogies between number theory and dynamical systems on foliated spaces. In particular, here is a quote from his paper *"On the nature of the 'explicit formulas' in analytic number theory"*:

> *Our example suggests that under suitable conditions transversal index theory [for smooth foliated manifolds] generalizes to solenoids or even more general laminated spaces instead of manifolds.*

In this talk, we discuss the nature of "general laminated spaces" and some of their properties, and give a selection of examples related to group actions and number theory.



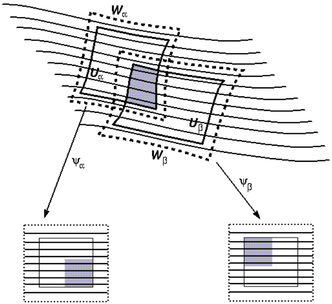
The objects of study are called various names in the literature:

- *Generalized laminations*, [Deninger, Ghys, Lyubich & Minsky]
- *Matchbox manifolds*, [Aarts & Martens, Clark & Hurder]
- *Solenoidal manifolds*, [Sullivan]

All are foliated spaces as introduced in the book

- Moore & Schochet, **Global analysis on foliated spaces**, 1988.

Definition: A C^r-foliation $\mathcal{F}$ of a manifold M is a "uniform partition" of M into submanifolds of constant dimension p and codimension q, such that there is a covering of M by C^r-coordinate charts whose change of coordinate functions preserve the leaves, for $r \geq 1$.



Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

- $\mathfrak{M}$ is a continuum $\equiv$ a compact, connected metric space;
- $\mathfrak{M}$ admits a covering by foliated coordinate charts
  $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$;
- each $\mathfrak{X}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$;
- plaques $\mathcal{P}_i(z) = \varphi_i^{-1}([-1,1]^n \times \{z\})$ are connected, $z \in \mathfrak{X}_i$;
- for $U_i \cap U_j \neq \emptyset$, each plaque $\mathcal{P}_i(z)$ intersects at most one plaque $\mathcal{P}_j(z')$, and changes of coordinates along intersection of plaques are smooth diffeomorphisms;

$+$ some other technicalities.

The path connected components of $\mathfrak{M}$ are the leaves of the foliation $\mathcal{F}$. To the above list, we add the condition:

• there is a leafwise smooth Riemannian metric on the leaves of $\mathcal{F}$, which is continuous in each foliation chart.

Proposition: Each leaf of $\mathcal{F}$ is a complete Riemannian manifold with bounded geometry.

It follows that associated to a matchbox manifold are many of the traditional aspects of Riemannian manifolds, such as leafwise curvature, leafwise De Rham cohomology, leafwise operators, foliation-preserving transformation groups, and invariants constructed from these data.

Basic question: What are these spaces? What are there properties and invariants?

Problems to investigate:

- Topological properties of laminated spaces.

  ⋆ Properties of minimal actions on Cantor spaces.

- Lefschetz Theorems for laminated spaces.

  ⋆ Extensions of Trace Formulae of Alvarez-Lopez and Kordyukov.

  ⋆ Distributional trace invariants associated to transverse profinite group actions.

- Relations to number theory.

  ⋆ Constructions of examples from absolute Galois groups.

The Lefschetz trace formulae of Alvarez-Lopez and Kordyukov appearing in Deninger's works are usually shown for flows on Riemannian foliations; that is, a smooth foliation of compact manifold with a holonomy-invariant transverse Riemannian metric.

The lamination analog of a Riemannian foliation of a compact manifold M, is a matchbox manifold $\mathfrak{M}$ whose holonomy is given by a minimal equicontinuous action on a Cantor space $\mathfrak{X}$:

- A Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in G.$$

The equicontinuous hypothesis has strong implications for the structure of $\mathfrak{M}$ as we next discuss. We make a brief detour, to introduce the notion of a *weak solenoid*.

Let M_0 be a connected closed manifold, and let $f^i_{i-1} : M_i \to M_{i-1}$ be a sequence of finite-to-one proper covering maps. Then

$$\begin{aligned} M_\infty &= \varprojlim \{ f^i_{i-1} : M_i \to M_{i-1} \mid i \geq 1 \} \qquad (1) \\ &= \{ (y_0, y_1, y_2, \ldots) \mid f^i_{i-1}(y_i) = y_{i-1} \mid i \geq 1 \} \end{aligned}$$

is a compact connected metrizable space called a *(weak) solenoid*.

There is a fibration map $\Pi_0 \colon M_\infty \to M_0$, and for $b \in M_0$ the fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$ is a Cantor space.

Theorem [McCord, 1966]. A solenoid is a matchbox manifold.

The classical *Vietoris solenoid* is an example.

Let $\mathbf{P} = (p_1, p_2, \ldots)$ be an infinite sequence, where each $p_i > 1$.

Let $f^i_{i-1} \colon \mathbb{S}^1 \to \mathbb{S}^1$ be a p_i-to-1 self-covering map of a circle.

A Vietoris solenoid is the inverse limit space

$$\Sigma_{\mathbf{P}} = \{(y_i) = (y_0, y_1, y_2, \ldots) \mid f^i_{i-1}(y_i) = y_{i-1} \ \} \subset \prod_{i \geq 0} \mathbb{S}^1$$

with subspace topology from the Tychonoff topology on $\prod_{i \geq 0} \mathbb{S}^1$.

Let $b \in \mathbb{S}^1$, then the fibre $\mathfrak{X}_b = \{(b, y_1, y_2, \ldots)\} \subset \Sigma_{\mathbf{P}}$ is a Cantor section, transverse to the foliation by path-connected components.

The fundamental group $\pi_1(\mathbb{S}^1, b) = \mathbb{Z}$ acts on $\mathfrak{X}_b$ via lifts of paths in $\mathbb{S}^1$, so the monodromy action on the fiber defines a group action $\Phi \colon \mathbb{Z} \times \mathfrak{X}_b \to \mathfrak{X}_b$ which is a classical odometer action.

The following result, whose proof is surprisingly technical, shows that the study of equicontinuous matchbox manifolds reduces to the study of weak solenoids:

Theorem [Clark & Hurder, 2013]. Let $\mathfrak{M}$ be a matchbox manifold with equicontinuous holonomy pseudogroup action. Then $\mathfrak{M}$ is foliated homeomorphic to a weak solenoid.

The study of weak solenoids reduces to the study of Cantor actions obtained from group chains. This approach was initiated by [Fokkink & Oversteegen, 2002].

Let $\Pi_0 \colon M_\infty \to M_0$ be a weak solenoid defined by the system of maps $\{f_0^i \colon M_i \to M_0 \mid i > 0\}$, where $f_0^i = f_0^1 \circ \cdots \circ f_{i-1}^i$.

Choose a basepoint $b \in M_0$ and basepoints $x_i \in M_i$ such that $f_0^i(x_i) = b$. Set $x = \lim x_i \in \mathfrak{X}_b \equiv \Pi_0^{-1}(b)$.

Define $G = G_0 = \pi_1(M_0, b)$, and let $G_i \subset G$ be the subgroup defined by $G_i = \mathrm{Image}\{(f_0^i)_\# \colon \pi_1(M_i, x_i) \to \pi_1(M_0, b)\}$.

$\{G_i \mid i \geq 0\}$ is a descending chain of subgroups of finite index in G.

The subgroups G_i are not assumed to be normal in G.

Example 1: Here is a simple example

Consider the Vietoris solenoid

$$\Sigma = \{f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1\}.$$

Then $G = \pi_1(\mathbb{Z}, 0) = \mathbb{Z}$, and $G_i = (p_1 \cdots p_i)\,\mathbb{Z}$, where p_i is the degree of f^i_{i-1}.

Since $\mathbb{Z}$ is abelian, each subgroup G_i is normal.

Each $X_i = G_0/G_i$ is a finite set with a left action of G. It is a group if G_i is normal in G. The Cantor fiber $\mathfrak{X}_b$ is identified with

$$\mathfrak{X}_b \cong \mathfrak{X}_\infty \equiv \varprojlim \; \{X_i \to X_{i-1}\} = \varprojlim \; \{G/G_i \to G/G_{i-1}\} .$$

and has a left G-action $\Phi\colon G \to \mathbf{Homeo}(X_\infty)$.

The action $(\mathfrak{X}_\infty, G, \Phi)$ is called a *generalized odometer*, or also called a *subodometer* by [Cortez & Petite, 2008].

The properties of a generalized odometer action are best seen by introducing the closure of the action, which is a profinite group.

Definition: The closure $E(\Phi)$ of $H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X}_\infty)$, in the topology of pointwise convergence on maps, is called the *Ellis (enveloping) semigroup*.

Proposition: Let Φ be an equicontinuous Cantor action. Then $E(\Phi) = \overline{H_\Phi}$ = closure of H_Φ in the *uniform topology on maps*.

For $x \in \mathfrak{X}_b$ let $\mathcal{D}_x = \{h \in \overline{H_\Phi} \mid h(x) = x\}$ be its isotropy group.

Lemma: The left action of $\overline{H_\Phi}$ on $\mathfrak{X}_\infty$ is transitive, hence $\mathfrak{X}_\infty \cong \overline{H_\Phi}/\mathcal{D}_x$ and the closed subgroup $\mathcal{D}_x \subset \overline{H_\Phi}$ is independent of the choice of basepoint x, up to topological isomorphism.

The normal core N of a subgroup $H \subset G$ is the largest subgroup $N \subset H$ which is normal in G.

Lemma: The normal core of $\mathcal{D}_x$ in $\overline{H_\Phi}$ is trivial.

Let $C_i \subset G_i$ be the normal core of G_i in G, then C_i has finite index in G. Define the profinite group

$$G_\infty \equiv \varprojlim \; \{q_i \colon G/C_{i+1} \to G/C_i \mid i > 0\} \; .$$

Each group G/C_i acts on the finite set $X_i = G/G_i$, so there is an induced action $\widehat{\Phi}_\infty \colon G_\infty \to \mathbf{Homeo}(\mathfrak{X}_\infty)$.

Theorem [Dyer-Hurder-Lukina, 2016]. $\overline{H_\Phi} = \widehat{\Phi}_\infty(G_\infty)$, and

$$\mathcal{D}_\infty \equiv \varprojlim \; \{\pi_i \colon G_{i+1}/C_{i+1} \to G_i/C_i \mid \ell \geq 0\} \; \cong \mathcal{D}_x \; . \qquad (2)$$

The group chain model for minimal equicontinuous Cantor actions is used to construct examples with prescribed special properties.

Example 1 continued:

For the Vietoris solenoid

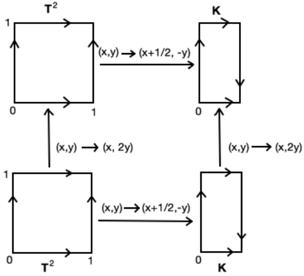
$$\Sigma = \{f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1\} \ ,$$

$G = \pi_1(\mathbb{Z}, 0) = \mathbb{Z}$, and $G/G_i = \mathbb{Z}/p_1 \cdots p_i\mathbb{Z}$, where p_i is the degree of f^i_{i-1}.

Since $\mathbb{Z}$ is abelian, $G_i = C_i$, and so G_i/C_i is a trivial group.

Thus $C_\infty \cong \mathfrak{X}_b$, where $\mathfrak{X}_b$ is a fibre of $\Sigma \to \mathbb{S}^1$, and so the discriminant group $\mathcal{D}_x$ of the Vietoris solenoid is trivial.

Example 2: Here is a more interesting example, with $\mathcal{D}_x$ non-trivial. It is due to [Rogers & Tollefson, 1971/72].



Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and consider an involution

$$r \times i(x,y) = (x + \tfrac{1}{2}, -y).$$

The quotient $K = \mathbb{T}^2/(x,y) \sim r \times i(x,y)$ is the Klein bottle.

The double cover $L\colon \mathbb{T}^2 \to \mathbb{T}^2\colon (x,y) \mapsto (x, 2y)$ induces a double cover $p : K \to K$.

Define K_∞ to be the inverse limit of the iterations of $p : K \to K$.

Since $i \circ L = p \circ i$, there is a double cover $i_\infty : \mathbb{T}_\infty \to K_\infty$.

Space K_∞ cannot be homogeneous.

The fundamental group of the Klein bottle is

$$G_0 = \pi_1(K, 0) = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

For the cover $p : K \to K$ we have

$$p_* \pi_1(K, 0) = \langle a^2, b \mid bab^{-1} = a^{-1} \rangle,$$

and for $p^n = p \circ \cdots \circ p : K \to K$ we have

$$G_n = (p^n)_* \pi_1(K, 0) = \langle a^{2^n}, b \mid bab^{-1} = a^{-1} \rangle.$$

The cosets of G/G_n are represented by $a^i G_i$, $i = 0, \ldots, n-1$,

$$C_n = \bigcap_{g \in G} g G_n g^{-1} = \langle a^{2^n} \mid bab^{-1} = a^{-1} \rangle.$$

Then $G_n / C_n = \{C_n, bC_n\}$, and so $\mathcal{D}_x \cong \mathbb{Z}/2\mathbb{Z}$.

We next discuss some properties of weak solenoids and their associated monodromy Cantor actions.

The *symmetries* of an action $(\mathfrak{X}_\infty, G, \Phi)$ are given by the right actions on $\mathfrak{X}_\infty$ that commute with the given left action.

These form a group, $\mathbf{Aut}(\mathfrak{X}_\infty, G, \Phi)$.

If $\mathfrak{X}_\infty$ is a group, then $\mathfrak{X}_\infty \cong G_\infty$ and the right action of G_∞ on $\mathfrak{X}_\infty$ is transitive.

Proposition: $\mathbf{Aut}(\mathfrak{X}_\infty, G, \Phi)$ is isomorphic to the commutant of $\mathcal{D}_x$ in G_∞. In particular, $\mathbf{Aut}(\mathfrak{X}_\infty, G, \Phi)$ acts transitively on $\mathfrak{X}_\infty$ if and only if $\mathcal{D}_x$ is the trivial group.

However, this is not the full story for symmetries of weak solenoids.

For a weak solenoid as defined by (1), and any $k \geq 1$ we have

$$M_\infty \cong \varprojlim \{ f^i_{i-1} : M_i \to M_{i-1} \mid i > k \} \tag{3}$$

Set $f^i_0 = f^1_0 \circ \cdots \circ f^i_{i-1}$. Then the fiber of the projection $\Pi_k \colon M_\infty \to M_k$ is the clopen set

$$\begin{aligned} \mathfrak{X}_k = \varprojlim \; \{ G_k/G_i \to G_k/G_{i-1} \mid i > k \} \\ = \; \{ (f^i_0(x_k), \ldots, x_k, y_{k+1}, \ldots) \in M_\infty \} \subset \mathfrak{X}_0 \; . \end{aligned}$$

The global monodromy with respect to Π_k is conjugate to the restricted action $\Phi_k \colon G_k \to \mathbf{Homeo}(\mathfrak{X}_k)$.

We derive the group quotient model for the action of G_k on $\mathfrak{X}_k$.

The key change is we must introduce the groups $C_{k,i} \subset G_k$, for $i > k$, which are the normal cores of G_i in G_k, so

$$C_{k,i} = \bigcap_{g \in G_k} g\, G_i\, g^{-1} \supset \bigcap_{g \in G} g\, G_i\, g^{-1} = C_{0,k} = C_k .$$

Then we have the collection of profinite groups

$$G_{k,\infty} \equiv \varprojlim \, \{q_i \colon G_k/C_{k,i+1} \to G_k/C_{k,i} \mid i > k\} \qquad (4)$$

$$C_{k,\infty} \equiv \varprojlim \, \{q_i \colon C_{k,i+1}/C_{i+1} \to C_{k,i}/C_i \mid i > k\} \qquad (5)$$

$$\mathcal{D}_{k,\infty} \equiv \varprojlim \, \{q_i \colon G_{i+1}/C_{k,i+1} \to G_i/C_{k,i} \mid i > k\} \qquad (6)$$

$$\mathcal{D}_{\infty} \equiv \varprojlim \, \{q_i \colon G_{i+1}/C_{i+1} \to G_i/C_i \mid i > k\} \qquad (7)$$

Each $G_{k,\infty}$ acts transitively on $\mathfrak{X}_k$ with isotropy group $\mathcal{D}_{k,\infty}$ at x. Moreover, there is an exact sequence

$$C_{k,\infty} \longrightarrow \mathcal{D}_\infty = \mathcal{D}_{0,\infty} \longrightarrow \mathcal{D}_{k,\infty}.$$

Also, for $\ell > k \geq 0$ there is a surjective map

$$\rho_{k,\ell} \colon \mathcal{D}_{k,\infty} \longrightarrow \mathcal{D}_{\ell,\infty} \tag{8}$$

and $C_{k,\infty} \subset C_{\ell,\infty}$.

The right action of $C_{k,\infty} \subset G_\infty$ on $X_\infty = G_\infty/\mathcal{D}_\infty$ fixes all the points in the clopen set $\mathfrak{X}_k$ but the action is not trivial on $\mathfrak{X}_\infty$.

Each group $C_{k,\infty}$ is an "internal symmetry" of the weak solenoid with monodromy is given by $(\mathfrak{X}_\infty, G, \Phi)$.

Definition: An action (X_∞, G, Φ) is *stable* if there exists $k_0 \geq 0$ such that for $\ell > k \geq k_0$ the map $\rho_{k,\ell}$ in (8) has trivial kernel.

The action is said to be *wild* otherwise.

Theorem [Hurder-Lukina, 2017]. Suppose that (X_∞, G, Φ) is a wild action. Then there exists a strictly increasing chain of indices $\{1 \leq k_1 < k_2 < \cdots\}$ so that the sequence of finite subgroups in the profinite group G_∞

$$C_{k_1,\infty} \subset C_{k_2,\infty} \subset \cdots \subset C_{k_j,\infty} \subset \cdots \subset \mathcal{D}_\infty$$

is strictly increasing.

Corollary: A weak solenoid with wild monodromy admits an infinitely increasing chain of internal symmetry groups.

We next discuss the notion of an "analytic Cantor action", which was introduced in the works of Alvarez Lopez, and its relation to wildness and the Hausdorff property for the action.

Let $U, V \subset \mathfrak{X}$ be clopen subsets of a Cantor space $\mathfrak{X}$.

• A homeomorphism $h \colon U \to V$ is quasi-analytic (QA) if either $U = V$ and h is the identity map, or for every *clopen* subset $W \subset U$ the fixed-point set of the restriction $h|W \colon W \to h(W) \subset V$ has no interior.

• A homeomorphism $h \colon U \to V$ is locally quasi-analytic (LQA) if for each $x \in U$ there exists a clopen neighborhood $x \in U' \subset U$ such that the restriction $h_{U'} \colon U' \to V' = H(U')$ is QA.

• A group action $\varphi \colon G \colon \mathfrak{X} \to \mathfrak{X}$ is LQA if for each $x \in \mathfrak{X}$, there exists a clopen neighborhood $x \in U$, such that the restrictions of elements of G to U are quasi-analytic.

Remarks:

• A free action $G \times \mathfrak{X} \to \mathfrak{X}$ is quasi-analytic.

• The automorphism group of a spherically homogeneous rooted tree T_d, acting on the Cantor set of ends, is not LQA.

Theorem [Hurder & Lukina, 2017]. Let $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action, where G is finitely generated. Then the action φ is stable if and only if the action of the profinite group G_∞ on X_∞ satisfies the LQA property.

The LQA property for a group action $(\mathfrak{X}, G, \Phi)$ can be interpreted in terms of the properties of the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ associated to the action.

Recall that for $g_1, g_2 \in G$, we say that $\Phi(g_1)$ and $\Phi(g_2)$ are *germinally equivalent* at $x \in \mathfrak{X}$ if $\Phi(g_1)(x) = \Phi(g_2)(x)$, and there exists an open neighborhood $x \in U \subset \mathfrak{X}$ such that the restrictions agree, $\Phi(g_1)|U = \Phi(g_2)|U$. We then write $\Phi(g_1) \sim_x \Phi(g_2)$.

For $g \in G$, denote the equivalence class of $\Phi(g)$ at x by $[g]_x$. The collection of germs $\mathcal{G}(\mathfrak{X}, G, \Phi) = \{[g]_x \mid g \in G \ , \ x \in \mathfrak{X}\}$ is given the sheaf topology, and forms an *étale groupoid* modeled on $\mathfrak{X}$.

Theorem [Hurder & Lukina, 2017]. If an action $(\mathfrak{X}, G, \Phi)$ is locally quasi-analytic, then $\mathcal{G}(\mathfrak{X}, G, \Phi)$ is Hausdorff.

The Hausdorff property for a germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \Phi)$ appears in the work of [Renault, 2008] on the C^*-algebra associated to the action, and has been studied in various works in C^*-algebras.

For the remainder of this talk, we discuss a variety of more subtle constructions of weak solenoids and Cantor actions, that illustrate all of the properties introduced above.

Example 3: 3-*dimensional matchbox manifolds.*

Let $\widetilde{M}_0 = \mathbb{H}$ be the real Heisenberg group, presented in the form $\mathbb{H} = (\mathbb{R}^3, *)$ with the group operation $*$ given by $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$. This operation is standard addition in the first two coordinates, and addition with a twist in the last coordinate. Let $\mathcal{H} = (\mathbb{Z}^3, *)$ be the integer lattice subgroup, so that $M_0 = \mathbb{H}/\mathcal{H}$ is a compact 3-manifold.

Consider subgroups of $\mathcal{H}$ which can be written in the form $\Gamma = M\mathbb{Z}^2 \times m\mathbb{Z}$ where $M = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is a 2-by-2 matrix with non-negative integer entries and $m > 0$ is an integer. Then $\gamma \in \Gamma$ is of the form $\gamma = (ix + jy, kx + ly, mz)$ for some $x, y, z \in \mathbb{Z}$. A straightforward computation gives the following:

Theorem [Dyer, 2015]. Let $A_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$, p and q are distinct primes. Define the group chain

$$G_0 = \mathcal{H} \quad , \quad \{G_n\}_{n\geq 1} = \{A_n\mathbb{Z}^2 \times p^n\mathbb{Z}\}_{n\geq 1}.$$

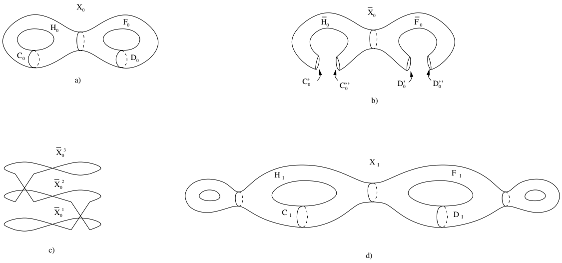
Then the discriminant $\mathcal{D}_\infty$ for the action is a Cantor group, and the action is stable.

Thus, the weak solenoid M_∞ defined by the coverings of M_0 associated to this chain is not homogeneous.

Note that the intersection $\bigcap_{n \geq 0} G_n = \{0\}$.

This implies that the leaf L_x of the foliation $\mathcal{F}$ on M_∞ through the basepoint $x \in \mathfrak{X}_b$ is isometric to the real Heisenberg group $\mathbb{H}$.

Example 4: [Schori, 1966] gave first example of a non-homogeneous weak solenoid. It is obtained by taking repeated 3-fold coverings starting with a closed surface Σ_2 of genus 2.



Proposition [Hurder & Lukina, 2017]. The monodromy action of $G = \pi_1(\Sigma_2, b_0)$ on the fiber of the solenoid over Σ_2 is wild.

Example 5: Wild actions of arithmetic lattices. Lubotzky [1993] showed that the profinite completions of higher rank arithmetic lattices contain arbitrary products of finite torsion groups.

$\mathbf{SL}_N(\mathbb{Z}) = N \times N$ matrices with integer entries and determinant 1

$\widehat{\mathbf{SL}_N(\mathbb{Z})}$ profinite completion of $\mathbf{SL}_N(\mathbb{Z})$

$\mathcal{P}$ = set of all primes

$$\widehat{\mathbf{SL}_N(\mathbb{Z})} \equiv \varprojlim \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z}) \cong \mathbf{SL}_N(\widehat{\mathbb{Z}}) \cong \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \ . \tag{9}$$

Let $G \subset \mathbf{SL}_N(\mathbb{Z})$ be a finite-index, torsion free subgroup.

Then G is finitely generated, and its profinite completion $\widehat{G}$ is a clopen subgroup of $\widehat{\mathbf{SL}_N(\mathbb{Z})}$, hence there is a cofinite $\mathcal{P}' \subset \mathcal{P}$, with

$$\prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \subset \widehat{G} \subset \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p)$$

Set $\widehat{H} = \prod_{p \in \mathcal{P}'} \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$. Then there is a homomorphism with dense image $\alpha \colon G \to \widehat{H}$. For each $p \in \mathcal{P}'$, choose $D_p \subset \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z})$ with trivial normal core. Set $\mathcal{D} = \prod_{p \in \mathcal{P}'} D_p$.

Theorem [Hurder & Lukina, 2017]. For a closed subgroup $\mathcal{D} \subset \widehat{H}$ as above, the induced action $\varphi_{\alpha,\mathcal{D}}$ of G on $\widehat{H}/\mathcal{D}$ by α satisfies:

- The action $\varphi_{\alpha,\mathcal{D}}$ is minimal and equicontinuous;
- The action $\varphi_{\alpha,\mathcal{D}}$ is wild for suitable choices of $\mathcal{D}$;
- The actions $\varphi_{\alpha,\mathcal{D}}$ for uncountably many such choices of $\mathcal{D}$ yield non-homeomorphic weak solenoids.

Example 6: Arboreal actions of Galois groups.

The analogy between theory of finite coverings and Galois theory of finite field extensions suggests looking for examples of minimal Cantor actions arising from purely arithmetic constructions.

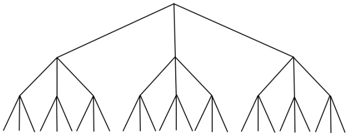
• [R.W.K. Odoni, 1985] began the study of arboreal representations of absolute Galois groups on the rooted trees formed by the solutions of iterated polynomial equations.

• [Jones, 2013] gives a nice introduction and survey of this program, from the point of view of arithmetic dynamical systems and number theory.

The following discussion concerns results of [Lukina, 2018].

Let $X = \mathcal{P}_d$ be the space of paths
in a spherically homogeneous rooted tree T_d.

Let G be any discrete group, acting on T_d
by permuting edges at each level
so that the paths are preserved.
The space of paths with the
cylinder topology is a Cantor set

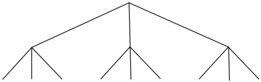


This action is equicontinuous.

Let $f(x)$ be an irreducible polynomial of degree d over a number field K. Let $\alpha \in K$, and suppose $f(x) = \alpha$ has d distinct solutions.

Identify α with the root of a d-ary tree T_d, and identify every solution $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1d}$ of $f(x) = \alpha$ with a vertex at level 1 in the tree.

$\mathrm{Gal}(K(f^{-1}(\alpha))/K)$ is identified with a subgroup of the symmetric group S_d.



For every α_{1i}, consider the equation

$$f(x) = \alpha_{1i}, \text{ so } f \circ f(x) = f(\alpha_{1i}) = \alpha.$$

Suppose there are d^2 distinct roots. Identify the solutions of $f(x) = \alpha_{1i}$ with the d vertices at level 2 connected with α_{1i} at level 1.

The action of $\mathrm{Gal}(K(f^{-2}(\alpha))/K)$ preserves the structure of the tree, so

$$\mathrm{Gal}(K(f^{-2}(\alpha))/K) \subseteq [S_d]^2,$$

where $[S_d]^2$ denotes the two-fold wreath product of symmetric groups S_d.

Continue by induction, assuming that for each $i > 0$ the polynomial $f^i(x)$ has d^i distinct roots.

In the limit, we get a d-ary infinite tree T_d of preimages of α under the iterations of $f(x)$, and the profinite group

$$\mathrm{Gal}_\infty(f) = \varprojlim \{\mathrm{Gal}(K(f^{-i}(\alpha))/K) \to \mathrm{Gal}(K(f^{-(i-1)}(\alpha))/K)\},$$

a subgroup of the infinite wreath product $Aut(T_d) = [S_d]^\infty$.

The group $\mathrm{Gal}_\infty(f)$ is called an *arboreal representation* of the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$.

The representation depends on the polynomial f.

Thus $\mathrm{Gal}_\infty(f)$ is a *profinite* group acting on the Cantor set of paths in the tree T_d.

Example [Odoni, 1985]. If $K = \mathbb{Q}$, $\alpha = 0$, $f(x) = x^2 - x + 1$, then

$$Gal_\infty(f) \cong Aut(T_2) \cong [S_2]^\infty \ .$$

Theorem [Lukina, 2018]. Let $f(x)$ be a polynomial of degree $d \geq 2$ over a field K, suppose all roots of $f^i(x)$ are distinct and $f^i(x) - \alpha$ is irreducible for all $i \geq 0$.

Let $\mathbf{v}$ be a path in the space of paths $\mathcal{P}_d$ of the tree T_d.

Then there exists a countably generated group G_0, a homomorphism $\Phi : G_0 \to Homeo(\mathcal{P}_d)$ and a chain $\{G_i\}_{i \geq 0}$ of subgroups in G_0 such that

(1) There is an isomorphism $\widetilde{\phi} : \overline{\Phi(G_0)} \to \mathrm{Gal}_\infty(f)$,

(2) There is a homeomorphism $\phi : \varprojlim\{G_0/G_i\} \to \mathcal{P}_d$ with $\phi(eG_i) = \mathbf{v}$,

(3) For all $\mathbf{u} \in \mathcal{P}_d$ and $\mathbf{g} \in \overline{\Phi(G_0)}$ we have

$$\widetilde{\phi}(\mathbf{g}) \cdot \phi(\mathbf{u}) = \phi(\mathbf{g}(\mathbf{u})).$$

Theorem [Lukina, 2018]. Suppose the image of an arboreal representation $\mathrm{Gal}_\infty(f)$ is a subgroup of finite index in $Aut(T_d)$. Then the action of the dense subgroup G_0 on the path space $\mathcal{P}_d$ is wild.

Remark: The proof of this result is geometric, it uses the absence of the *strong quasi-analytic property* of wild actions.

The proof does not require an explicit description of the Galois groups involved.

Theorem [Lukina, 2018]. Let p and d be distinct odd primes, let $K = \mathbb{Q}_p$ be the field of p-adic numbers, and let

$$f(x) = (x + p)^d - p.$$

Then the action of the dense subgroup group G_0 of the arboreal representation $Gal_\infty(f)$ is stable.

Remark: The group acting on T_d in this result is the Baumslag-Solitar group $BS(p,1) = \{\tau, \sigma \mid \sigma\tau\sigma^{-1} = \tau^p\}$. The proof uses explicit representations of subgroups G_i in terms of generators and relations.

There are many further constructions of arboreal actions, by [Boston & Jones, 2007] and Nekrashevych for example. It is work in progress, to understand further the structure of arboreal Galois actions, from the point of view of the geometry and dynamics of the weak solenoids associated to these actions.

Conclusions:

• The construction of weak solenoids whose monodromy action has interesting properties, such as non-trivial Cantor discriminant group and possibly also wild, is closely related to constructions in geometric group theory and number theory.

• The invariants of weak solenoids obtained using foliation index theorems and Lefschetz trace formulae, are as good a place as any, to go hunting for number theory formulae...

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Happy
Birthday