Renormalization and dimension for Kuperberg minimal sets

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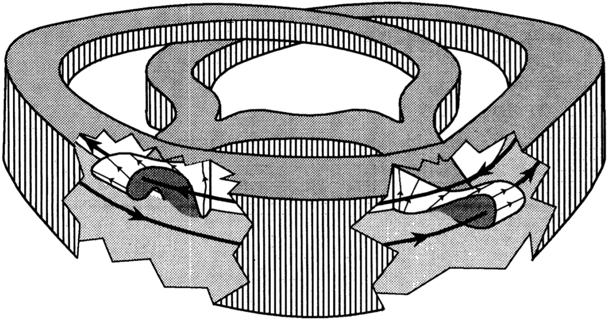
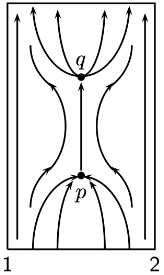
Kuperberg's Theorem

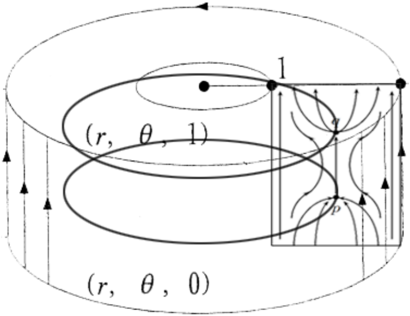
Theorem: [Kuperberg, 1994] Let M be a closed 3-manifold M. Then there exists a smooth, non-vanishing vector $\mathcal{X}$ on M with no periodic orbits.



Wilson Plug

Wilson's fundamental idea was the construction of a plug which trapped content, and all trapped orbits have limit set a periodic orbit contained in the plug. The two periodic orbits are attractors:





Schweitzer's Theorem

Theorem: [Schweitzer, 1974] Every homotopy class of non-singular vector fields on a closed 3-manifold M contains a C^1-vector field with no closed orbits.

In early 1970's, Paul Schweitzer had two deep insights:

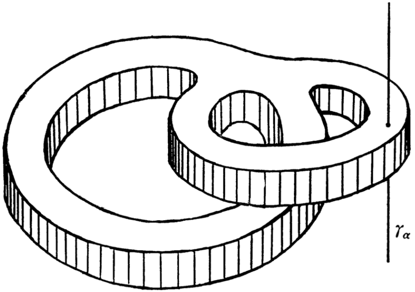
In the Wilson Plug:

• the periodic circles can be replaced by a minimal set for a flow without periodic orbits, such as the Denjoy minimal set;

• the new minimal set does not have to be in a planar flow, but may be contained in a surface flow which embeds in $\mathbb{R}^3$.

Schweitzer Plug

The circular orbits of the Wilson Plug are replaced by Denjoy minimal sets, embedded as follows:



Kuperberg's "Big Idea"

Shigenori Matsumoto's summary:

そこで，どうしても W 内のふたつの周期軌道 T_1 と T_2 を予め破壊しておく必要がある．しかしそのために新しい部品を開発するのでは話は振り出しに戻ってしまう．Kuperberg の発想は，W 内の**周期軌道自身で自分達を破壊させる**というものである．敵同士が妨害工作をしあうようにわなを仕掛けた後は，何もせずに黙ってみていればよいということである．

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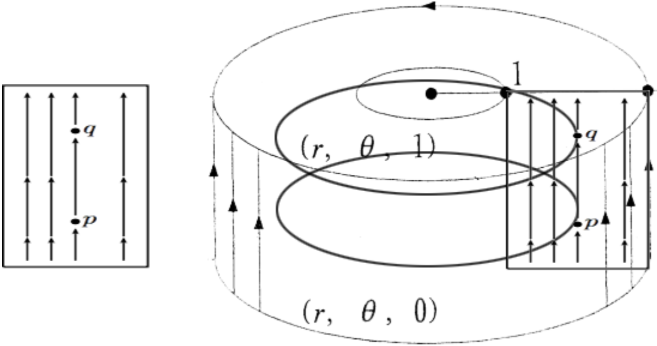
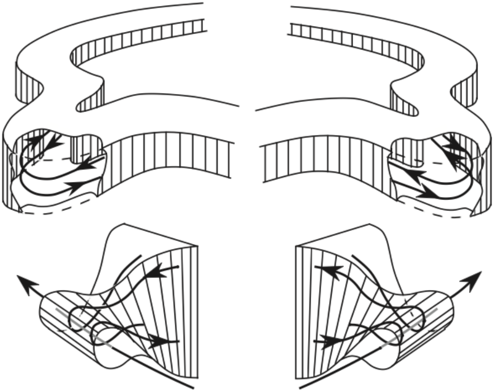
> *We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to let closed orbits demolish themselves. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.*
>
> *(transl. by Kiki Hudson Arai)*

Modified Wilson Plug

The vector field $\mathcal{W} = f(r,z)\partial/\partial z + g(r,z)\partial/\partial\theta$ on the plug $(r,\theta,z) \in [1,3] \times \mathbb{S}^1 \times [-2,2] = W$ is radially symmetric, with $f(r,0) = 0$ and $g(r,z) = 0$ only near the boundary, and $g(r,z) = 1$ away from the boundary ∂W.

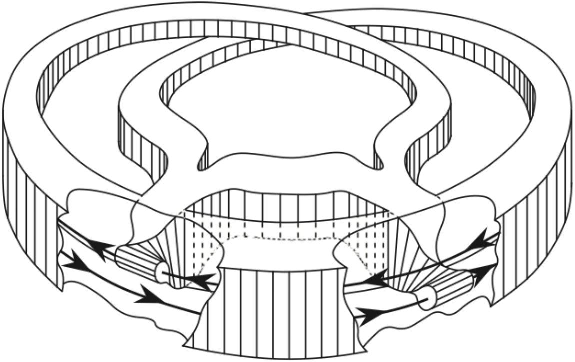


Form and twist the horns



Insert the twisted horns

The vector field $\mathcal{W}$ induces a field $\mathcal{K}$ on the surgered manifold.



Minimal set for the Kuperberg Plug

If $x \in P$ has forward orbit trapped, then the ω-limit set ω_x is a closed invariant subset of the interior of P, hence contains a minimal set K for the flow.

Theorem: The Kuperberg flow has a unique minimal set Σ, given by the closure of the orbit of the image of a *special point* from either of the periodic orbits in the Wilson Plug.

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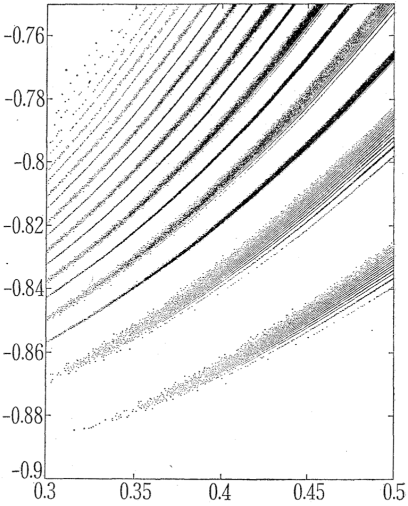
Problem: Describe the dynamics of $\mathcal{X}$ restricted to Σ.

Problem: What is the topological shape of Σ?

Computer model of K



Zippered laminations

Theorem:[H & R] The Kuperberg flow preserves a “zippered lamination” $\mathfrak{M}$, where:

• $\mathfrak{M}$ is defined by the closure of the orbit of the Reeb Cylinder.

• The boundary $\mathcal{Z} = \partial\mathfrak{M}$ is the flow of a Cantor set (the *zipper*), defined by 2-dimensional ping-pong dynamics.

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Theorem:[H & R] The minimal set Σ has *unstable shape*.

The minimal set

Theorem:[H & R] $\Sigma = \mathfrak{M}$ for a "generic" Kuperberg Plug.

By generic, we mean that the singularities for the vanishing of the vertical vector field $\mathcal{W}$ used to define $\mathcal{K}$ are quadratic type, and the insertion yields a quadratic radius function.

The two papers below contain particular constructions of Kuperberg Plugs which discuss the existence of open disks in Σ.

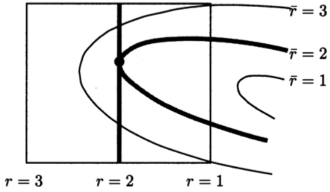
• É. Ghys, "Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)", Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, 1995.

• G. Kuperberg and K. Kuperberg, "Generalized counterexamples to the Seifert conjecture" , Annals of Math, 1996.

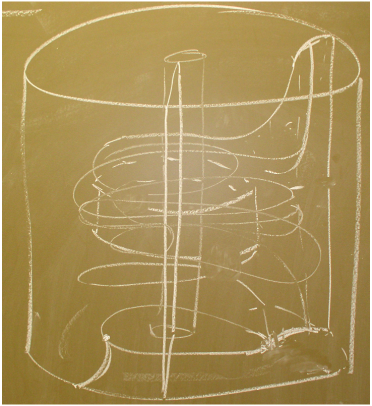
Families of orbits

Strategy is to consider the orbits of transversal segments to the Kuperberg flow, obtaining global dynamical information as well.

Radius inequality: $\bar{r}(x) \geq r(x)$, where $\bar{r}(x)$ is the radius of the image of x in the inserted region, with equality only at the periodic orbit entry point. Thus, passing through the face of an insertion increases the radius.

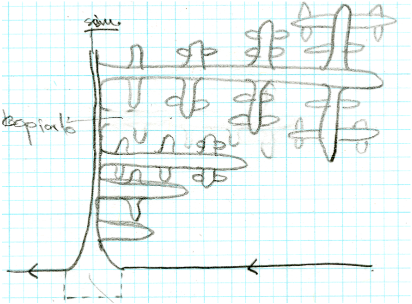


Tongues in Wilson Plug



Propellors in Kuperberg Plug

When a tongue as above is combined with the recursive Wilson dynamics of the Kuperberg flow, we obtain a “propellor”:



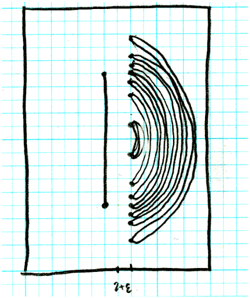
Propellors produce leaves of the lamination

The orbit of the Reeb cylinder $\mathcal{R} = \{-1 \leq z \leq 1\}$ produces arcs:

Ellipses capture full dynamics

The orbit of the full cylinder $\mathcal{C} = \{-2 \leq z \leq 2\}$ produces ellipses.

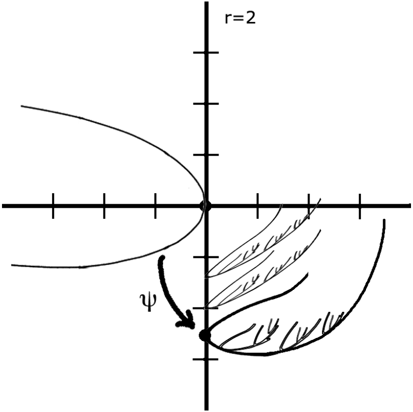
Going to tip of propellor $\equiv$ moving toward left most ellipse.



Re-insertion of the propellors

Dynamics are *renormalizable*, so nesting within nesting ...

Left arcs are lamination $\mathfrak{M}$. Right arcs are recurrence sets.

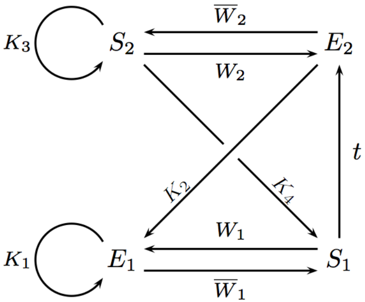


Coding

The pseudogroup dynamics of the flow admits a coding.

E_i are entry faces of insertions, S_i are exit faces of insertions.

Orbits in Σ have quasi-tiling property: "tiles" are the codes for Wilson orbit segments.



Wilson sub-dynamics

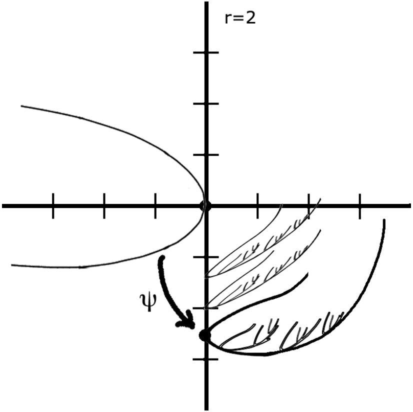
Let Φ_t denote flow of $\mathcal{K}$. Let p_1 be special point - image of Wilson periodic orbit in transversal E_1.

The there exists times $0 < s_1 < s_2 < \cdots$ with $s_\ell \to \infty$ very rapidly, $\xi_\ell = \Phi_{s_\ell}(p_1)$ with, for (r, θ, z) coordinates on E_1

- radius $r(\xi_\ell) = r(p_1)$ constant,
- angle $\theta(\xi_\ell) = \theta(\xi_1)$ constant;
- height $z(\xi_\ell)$ monotone increasing to $-1 = z(p_1)$.
- $|z(\xi_{\ell+1}) - z(\xi_\ell)|$ decreases polynomial speed as $\ell \to \infty$.

Re-insertion dynamics

$K_1 \colon E_1 \to E_1$ is reinsertion map. $\psi_i = \Phi_{s_{\ell_i}} \circ K_1 \circ \Phi_{s_{\ell_0}}$ yields



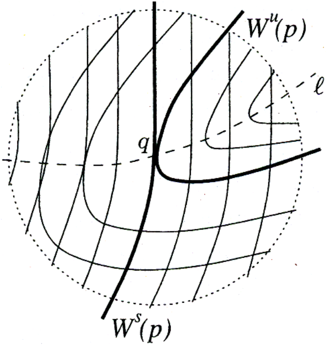
Ping Pong dynamics

Pseudogroup generated by $\langle \psi_i, \psi_j \rangle$ for $i \neq j$ large generates "free" semi-sub-pseudogroup.

- The problem is that the domains are planar, not linear.
- Orbits are chaotic in open neighborhood of special point p_1 - the renormalization point.
- Chaos is seen in tips of propellors.

Almost homoclinic dynamics

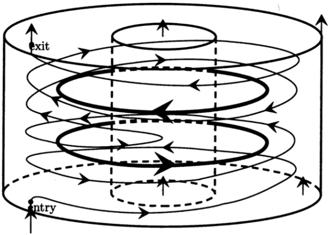
The dynamics generated by ψ_i maps resemble that of homoclinic tangencies of partially hyperbolic systems:



Reeb circle

The circle $\mathcal{S} = \{(r, \theta, z) \mid \ r = 2\ ,\ z = 0\}$ is called Reeb circle.

- Wilson orbit for x with $0 < |r(x) - 2| < \epsilon$ traverses $\mathcal{S}$ within ϵ.
- Wilson flow is rotation on circles $\{(r, \theta, z) \mid \ r \neq 2,\ z = cst\}$.
- Flow of $\mathcal{S}$ under Φ_t defines lamination $\mathfrak{M}$ (up to closure).



Density

Proposition: The Φ_t-orbits of special point is dense in an open tubular neighbor hood of $\mathcal{S}$.

Proof (by G. & K. Kuperberg): Choose vector field $\mathcal{K}$ precisely, then show explicitly.

Proof (by É. Ghys): Choose vector field $\mathcal{K}$ carefully, then calculate and invoke Baire for subset of insertions.

Proof (by H. & R.): Choose vector field $\mathcal{K}$ and insertions generically, then special orbits $\{\xi_\ell\}$ have polynomial density, and follow image of this set under renormalization to get density in arbitrary open neighborhoods of points in the Reeb circle $\mathcal{S}$.

Bibliography

• F.W. Wilson, "On the minimal sets of non-singular vector fields", Annals of Math, 1966.

• P. Schweitzer, "Counterexamples to the Seifert conjecture and opening closed leaves of foliations", Annals of Math, 1974.

• K. Kuperberg, "A smooth counterexample to the Seifert conjecture", Annals of Math, 1994.

• G. Kuperberg & K. Kuperberg, "Generalized counterexamples to the Seifert conjecture" , Annals of Math, 1996.

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• S. Matsumoto, "Kuperberg's C^∞ counterexample to the Seifert conjecture", Sūgaku, Mathematical Society of Japan, 1995.

• S. Hurder & A. Rechtman, "Dynamics and topology of the Kuperberg minimal set", 2011.