Dynamics and topology of matchbox manifolds

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Introduction

We present an approach to the study of the topology, dynamics and spectral geometry of foliations, based on studying special types of "cycles", which combine "classical" ideas in the subject:

- Minimal sets for dynamical systems
- Classification of Cantor pseudo-group actions
- Adherence properties of leafwise spectrum
- Spectral flow and index

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First, we ask . . .

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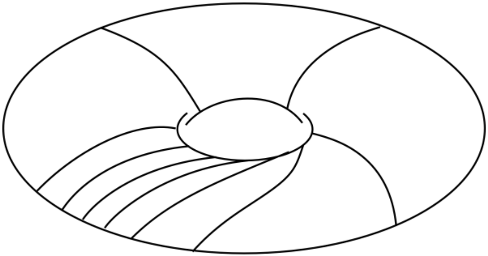

$\mathcal{F}$ foliates a smooth manifold, the 2-torus $\mathbb{T}^2$.

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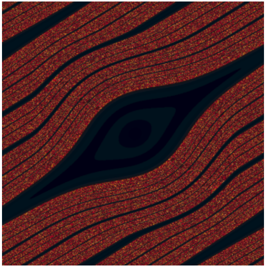

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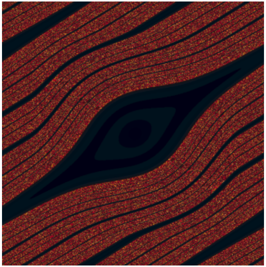

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A typical foliation exhibits "chaos" and other types of "mixing" of its leaves, so these examples are atypical.

What is a . . . leaf?

A foliation is assembled from its leaves.

Problem: Let L be a complete Riemannian manifold. When is L *homeomorphic*, possibly by a *quasi-isometry*, to a leaf of some foliation? Do we know which open manifolds are leaves?

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• Codimension $q = 1$, there are some non-leaf results:

Ghys; Inaba, Nishimori, Takamura, Tsuchiya [1984-85].

• Codimension $q > 1$, average Euler class, Phillips & Sullivan [1981]; average Pontrjagin classes, Januszkiewicz [1984].

• Index theory on open manifolds, thesis of Roe [1988], possibly give examples of non-leaves using spectral geometry.

What is a . . . foliation characteristic class?

An *invariant* of a foliation, to distinguish one from another.

A "foliation characteristic class" is a cohomology (or homotopy, or scalar) invariant associated to the tangent bundle to the leaves of $\mathcal{F}$, or to the normal bundle to $\mathcal{F}$, or possibly something else . . .

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Problem: What aspect of a foliation do these classes depend on? What do they "measure"?

The study of the Euler and secondary classes of foliations suggest properties of foliations to investigate further.

What is a . . . foliation C^*-algebra?

A *analytic model* of a foliation, analogous to classifying spaces.

$C_r^*(M/\mathcal{F})$ is a non-commutative C^*-algebra associated to fields of compact operators along the leaves of $\mathcal{F}$.

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Problem: How do you recover the "structure" of a foliation from its C^*-algebra? What geometric and dynamical properties does it determine? Can you determine from the properties of $C^*_r(M/\mathcal{F})$ if the foliation has non-zero entropy, for example?

These questions have been central to the study of foliation theory for the past 30 or 40 years. Answers have really only been given in special classes of foliations:

- $\mathcal{F}$ is transversally Riemannian;
- $\mathcal{F}$ is transversally affine;
- $\mathcal{F}$ defined by free action of Lie group;
- $\mathcal{F}$ has codimension-one on closed manifold M;
- $\mathcal{F}$ is foliation on M with dimension 2 or 3.

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A new approach in the subject is to consider these questions and classes of foliations in a more general setting, and develop additional techniques for understanding.

Closed invariant sets

Every dynamical system on a compact space has a collection of closed invariant sets.

$L \subset M$ a leaf of $\mathcal{F}$ in compact manifold, its closure $X = \overline{L}$ is closed and invariant. The minimal sets $\mathfrak{M} \subset \overline{L}$ have special significance.

Question: *Generalized Poincaré Recurrence principle:* dynamical properties of L and topological properties of $\mathfrak{M}$ are closely related.

• *How* is not at all clear, especially for codimension $q > 1$.

• In classical dynamical systems, role of closed invariant sets is fundamental.

Generalized foliation cycles

Idea: In foliation theory, the minimal sets are embedded "generalized foliation cycles", for all aspects of their study:

- Describe the "generalized foliation cycles";
- Study the ways in which cycles can be embedded in foliations;
- Find implications for dynamics, cohomology and spectral geometry of foliations, using these embedded cycles.

Foliated spaces

Definition: $\mathfrak{M}$ is an *n-dimensional foliated space* if:

$\mathfrak{M}$ is a compact metrizable space, and each $x \in \mathfrak{M}$ has an open neighborhood homeomorphic to $(-1,1)^n \times \mathfrak{T}_x$, where $\mathfrak{T}_x$ is a closed subset with interior of some Polish space $\mathfrak{X}$.

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A foliated space $\mathfrak{M}$ can be a usual foliated manifold, or at the other extreme, a foliation of codimension zero.

Matchbox manifolds

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

• $\mathfrak{M}$ is a continuum, a compact, connected metrizable space;

• $\mathfrak{M}$ admits a covering by foliated coordinate charts
$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$;

• each $\mathfrak{T}_i$ is a clopen subset of a totally disconnected space $\mathfrak{X}$.

Then the arc-components of $\mathfrak{M}$ are locally Euclidean:

$$\mathfrak{T}_i \quad \text{are totally disconnected} \iff \mathfrak{M} \quad \text{is a matchbox manifold}$$

A “smooth matchbox manifold” $\mathfrak{M}$ is analogous to a compact manifold, and the pseudogroup dynamics of the foliation $\mathcal{F}$ on the transverse fibers $\mathfrak{T}_i$ represents *intrinsic* fundamental groupoid.

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The "matchbox manifold" concept is much more general than minimal sets for foliations: they also appear in study of tiling spaces, subshifts of finite type, graph constructions, generalized solenoids, pseudogroup actions on totally disconnected spaces, . . .

Theme of research is that this class of foliated spaces is *universal* in an appropriate sense.

Examples from dynamical systems

Minimal $\mathbb{Z}^n$-actions on Cantor set K or symbolic space:

- Adding machines (minimal equicontinuous systems)
- Toeplitz subshifts over $\mathbb{Z}^n$
- Minimal subshifts over $\mathbb{Z}^n$
- Sturmian subshifts

All of these examples are realized as Cantor bundles over base $\mathbb{T}^n$.

More examples

Replace $\mathbb{Z}^n$ by an finitely generated group Γ, the torus $\mathbb{T}^n$ by a compact manifold B with $\pi_1(B, b_0) \cong \Gamma$, choose a transitive action of Γ on a Cantor set, and form the suspension foliation.

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First part of program is to try to understand what types of examples can be analyzed in this class of foliated spaces.

We consider some very classical examples.

Vietoris solenoids

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$$\xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \mathbb{S}^1 \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

Then the p_ℓ are the *bonding maps* of degree $p_\ell > 1$ for the solenoid

$$\mathcal{S} = \lim_{\leftarrow} \, \{p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell=0}^{\infty} \mathbb{S}^1$$

Proposition: $\mathcal{S}$ is a continuum with an equicontinuous flow $\mathcal{F}$, so is a 1-dimensional matchbox manifold.

Weak solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving proper covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \lim_{\longleftarrow} \, \{p_\ell \colon B_\ell \to B_{\ell-1}\} \subset \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: $\mathcal{S}$ has natural structure of a matchbox manifold, with every leaf dense.

The dynamics of $\mathcal{F}$ are *equicontinuous*. Though, the geometry of the leaves in this class of examples is not well-understood.

McCord solenoids

Basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

Definition: A weak solenoid $\mathcal{S}$ is a *McCord solenoid*, if for some fixed $\ell_0 \geq 0$, then for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}.

Classifying weak solenoids

A weak solenoid is "determined" by the base manifold B_0 and the tower equivalence of the inverse chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

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Theorem: [Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^k$ with $k \geq 2$, the homeomorphism types of McCord solenoids are not classifiable, *in the sense of Descriptive Set Theory*.

The number of such is not just huge, but indescribably large.

More examples from graph closures

The Ghys-Kenyon examples are obtained from the closure of space of subtrees of given graph.

This yields a yields a Cantor set with pseudogroup action, which generates a matchbox manifold $\mathfrak{M}$, with *expansive* dynamics. These examples relate graph geometry with matchbox manifold dynamics, and are the opposite of solenoids.

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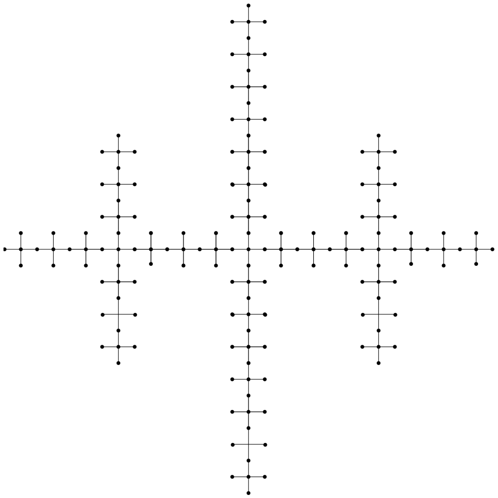
- Ghys [1999]
- Blanc [2001]
- Lozano-Rojo [2005]
- Lukina [2011]
- Lozano-Rojo & Lukina [2012].

A sample graph



Embeddings

The examples above correspond to various aspects of the study of smooth foliated manifolds:

- perturbations of foliations
- local orbit behavior near leaves with holonomy
- graphing the equivalence relation of a foliation

A matchbox manifold is a type of "discrete approximation" to a smooth foliation.

Universality Properties

Embedding Property: Given a matchbox manifold $\mathfrak{M}$, does there exists a C^r-foliation $\mathcal{F}_M$ of a compact manifold M and an embedding of $\iota \colon \mathfrak{M} \hookrightarrow M$ as a foliated subspace? $(r \geq 0)$

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Resolution Property: Given a minimal set $\mathcal{Z} \subset M$ in a foliated manifold M, does there exists a matchbox manifold $\mathfrak{M}$ and a foliated quotient map $\pi\colon \mathfrak{M} \to \mathcal{Z}$ which maps leaves to leaves homeomorphically?

Embeddings

Solutions to the embedding problem for solenoids modeled on $\mathbb{S}^1$ were given by Gambaudo, Tressier, et al in 1990's.

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The criteria for embedding depends on the degree of smoothness required, and the tower of subgroups of the fundamental group.

Instability

One application is a type of “Reeb Instability” result:

Theorem: [Clark & H 2010] Let $\mathcal{F}_0$ be a C^∞-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some saturated open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}_M$ on M which is C^∞-close to $\mathcal{F}_0$, and $\mathcal{F}_M$ has an uncountable set of solenoidal minimal sets $\{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$, which are *pairwise non-homeomorphic*.

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Solenoid-type objects are "typical" for perturbations of dynamical systems and possibly also for foliations, so study them to understand general problems about foliations.

Pseudogroups

Dynamics of $\mathfrak{M}$ defined by a *pseudogroup* action on a Cantor set.

Covering of $\mathfrak{M}$ by foliation charts $\Longrightarrow$ transversal $\mathcal{T} \subset \mathfrak{M}$ for $\mathcal{F}$

Holonomy of $\mathcal{F}$ on $\mathcal{T}$ $\Longrightarrow$ compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$:

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ such that $\langle \Gamma \rangle = \mathcal{G}_{\mathcal{F}} | \mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$,
  $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Topological dynamics

Dynamical properties of $\mathcal{F}$ formulated in terms of $\mathcal{G}_{\mathcal{F}}$; e.g.,

$\mathcal{F}$ has no leafwise holonomy if for $g \in \mathcal{G}_{\mathcal{F}}$, $x \in Dom(g)$, $g(x) = x$ implies $g|V = Id$ for some open neighborhood $x \in V \subset \mathcal{T}$.

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Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \implies d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

Equicontinuous matchbox manifolds

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We have a "Molino Theorem" for matchbox manifolds:

Theorem: [Clark & H 2011] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathcal{F}$ is minimal, and

• $\mathfrak{M}$ homeomorphic to a weak solenoid, so is homeomorphic to the suspension of an minimal equicontinuous action of a countable group on a Cantor space $\mathbb{K}$.

• $\mathfrak{M}$ homogeneous $\Longrightarrow$ homeomorphic to a McCord solenoid

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We know "what they are", but cannot classify - just too many.

Expansive matchbox manifolds

Definition: $\mathfrak{M}$ is an *expansive matchbox manifold* if it admits some covering by foliation charts, and there exists $\epsilon > 0$, so that for all $w \neq w' \in \mathcal{T}_i$ for some $1 \leq i \leq k$ with $d_{\mathcal{T}}(w, w') < \epsilon$, then there exists a leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ starting at w and ending at some $z \in \mathcal{T}$ with $w, w' \in \mathrm{Dom}(h_{\tau_{w,z}})$ such that $d_{\mathcal{T}}(h_{\tau_{w,z}}(w), h_{\tau_{w,z}}(w')) \geq \epsilon$.

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Example: The Denjoy minimal set is expansive.

Example: Let Δ be a *quasi-periodic tiling* of $\mathbb{R}^n$, which is *repetitive*, *aperiodic*, and has *finite local complexity*, then the "hull closure" Ω_Δ of the translates of Δ by the action of $\mathbb{R}^n$ defines an expansive matchbox manifold $\mathfrak{M}$.

Generalized solenoids

A generalized solenoid is defined by a tower of *branched manifolds*

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The *bonding maps* p_ℓ are assumed to be locally smooth cellular maps, but not necessarily covering maps. Set:

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Proposition: If the local degrees of the maps p_ℓ tend to ∞, then the inverse limit $\mathcal{S}$ has natural structure of a matchbox manifold.

These are more general than the *Williams solenoids*, but same idea.

Theorem: (Anderson & Putnam [1991], Sadun [2003]) A *quasi-periodic tiling* Δ of $\mathbb{R}^n$, which is *repetitive*, *aperiodic*, and has *finite local complexity*, then the "hull closure" Ω_Δ of the translates of Δ by the action of $\mathbb{R}^n$ is homeomorphic to a generalized solenoid.

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This approximation theory is one key to analyzing the spectrum of operators for quasi-periodic tilings.

Theorem: [Clark, H, Lukina 2011] Let $\mathfrak{M}$ be a minimal expansive matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a generalized solenoid.

This is a result about the *shape theory* of matchbox manifolds.

For a minimal set in codimension-one foliation, this coincides with what we know about the shape of these exceptional minimal sets.

It implies that expansive foliations admit a type of generalized coding for their dynamics, where the leaves of $\mathcal{F}$ are approximated by compact manifolds with boundary.

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- Equicontinuous case corresponds to smooth Riemannian foliations: admit structure theory, modeled on homogeneous Cantor spaces.

- Matchbox manifolds: special class of “discrete” foliated spaces.

- Represent approximations to smooth foliations.

- Dynamics separates cases: equicontinuous verses expansive.

- Equicontinuous case corresponds to smooth Riemannian foliations: admit structure theory, modeled on homogeneous Cantor spaces.

- Expansive case corresponds to chaotic foliations: admit structure theory, modeled on generalized coding for Cantor spaces.

- Expansive case is not at all well-understood: when are they represented as almost 1-1 factors?

Spectrum of foliations

In the next lecture, we consider the spectrum of leafwise elliptic operators associated to matchbox manifolds, and how the properties discussed above influence the spectrum and the associated foliation C^*-algebras.

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Thank you for your time and attention.

http://www.math.uic.edu/∼hurder/talks/Nagoya20120321.pdf