Spectral Cycles for Foliations

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We consider some aspects of how foliation spectral geometry is related to the generalized foliation cycles introduced in Lecture 1.

M is a closed Riemannian manifold.

$\mathcal{F}_M$ is an n-dimensional foliation on M.

$\mathcal{E} \to M$ an Hermitian bundle, for which there is a first-order geometric operator along the leaves of $\mathcal{F}$,

$$\mathcal{D} \colon C^\infty(\mathcal{F}, \mathcal{E}) \to C^\infty(\mathcal{F}, \mathcal{E})$$

That is, for each leaf L of $\mathcal{F}_M$ the restriction

$$\mathcal{D}_L \colon C_c^\infty(L, \mathcal{E}_L) \to C_c^\infty(L, \mathcal{E}_L)$$

is elliptic and essentially self-adjoint.

For each leaf L of $\mathcal{F}$, the restricted operator

$$\mathcal{D}_L \colon C_c^\infty(L, \mathcal{E}_L) \to C_c^\infty(L, \mathcal{E}_L)$$

is elliptic and essentially self-adjoint, so we can define its spectrum, and the decomposition into pure and continuous parts:

$$\sigma(\mathcal{D}_L) = \sigma_a(\mathcal{D}_L) \cup \sigma_c(\mathcal{D}_L) \subset \mathbb{R}$$

In general, very little is known about this. For related discussions:

Topology of covers and the spectral theory of geometric operators, in *Index theory and operator algebras (Boulder 1991)*, Contemp. Math. Vol. 148, Amer. Math. Soc., 1993.

Gap Labeling & K-Theory

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Another special case is based on the spectral flow approach via odd index theory for foliations, as in the solutions of Bellisard's "Gap Labeling Conjecture" in the study of quasi-crystals on $\mathbb{R}^n$:

- Kaminker & Putnam [2003]
- Benameur & Oyono-Oyono [2003]
- Bellissard, Benedetti & Gambaudo [2006]

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The idea of applying the generalized foliation cycles to the study of leafwise spectrum, is to convert every such problem into a "gap labeling" question, or at least one involving spectral flow of geometric operators.

KK-Index

$C^*(M/\mathcal{F}_M)$ is the *reduced* C^*-algebra associated to $\mathcal{F}_M$.

Theorem: [Connes-Fack 1984] A first-order geometric operator $\mathcal{D}$ along the leaves of $\mathcal{F}$ defines a generalized index class

$$[\mathcal{D}] \in KK(M, M/\mathcal{F}) \cong KK(C(M), C^*(M/\mathcal{F}))$$

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Given a K-Theory class $[E] \in K(M)$ we obtain the "index class"

$$[E] \cap [\mathcal{D}] \in K^*(C^*(M/\mathcal{F}))$$

Some basic problems

The pairing $[E] \cap [\mathcal{D}]$ is called "integration along the leaves" of $\mathcal{F}$, of the K-theory class $[E]$ against the *leafwise fundamental class* defined by $\mathcal{D}$, to obtain the non-commutative index class

$$[E] \cap [\mathcal{D}] = \mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E) \in K^*(C^*(M/\mathcal{F}))$$

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Two basic problems in *foliation index theory*:

- What is the *analytic* meaning of $\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E)$?
- How does $\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E)$ depend on the geometry/dynamics of $\mathcal{F}$?

Some known relations

- The Lichnerowicz formula implies vanishing of components of the index class $[E] \cap [\mathcal{D}]$ when the leaves of $\mathcal{F}$ have positive scalar curvature.

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• When the leaves of $\mathcal{F}$ have holonomy covers which satisfy the *coarse Baum-Connes Conjecture*, then pairing with $[\mathcal{D}]$ is a faithful.

See the papers:

Exotic index theory for foliations, *preprint*, 1993.

Coarse geometry of foliations, in *Geometric study of foliations (Tokyo, 1993)*, World Sci. Publishing, 1994.

Transverse measures and traces

Let Λ be a holonomy-invariant transverse measure for the foliation $\mathcal{F}$. That is, for a Borel subset $E \subset M$ which intersects each leaf L at most countable many times, then $\Lambda(E) \in \mathbb{R}$ is invariant under the holonomy translations of $\mathcal{F}$.

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Associated to Λ is a trace $Tr_\Lambda \colon C^*(M/\mathcal{F}) \to \mathbb{R}$.

Proposition: The Λ-dimension function $Tr_\Lambda \colon K^0(C^*(M/\mathcal{F})) \to \mathbb{R}$ is well-defined, and measures the differences of the *von Neumann dimensions* of the projections in $C^*(M/\mathcal{F})$ defining a class.

Transverse measures and foliated spaces

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Problem: What is the analytic meaning of the restricted index? How does it depend on the dynamics of $\mathcal{F}|\mathcal{Z}(\Lambda)$?

Generic invariant sets

Let $\mathcal{Z} \subset M$ be a closed saturated subset, with foliation $\mathcal{F}$.

Definition: $\mathcal{Z}$ is *generic* if each leaf $L \subset \mathfrak{M}$ without holonomy for $\mathcal{F}$, is also without holonomy as a leaf for $\mathcal{F}$ on M.

If the leaves of $\mathcal{F}$ are all simply connected, then $\mathcal{Z}$ is generic.

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Proposition: [Fack & Skandalis 1982] If $\mathcal{Z} \subset M$ is a generic closed invariant set, then there is a well-defined restriction map

$$\iota_{M,\mathcal{Z}} \colon C^*(M/\mathcal{F}) \to C^*(\mathcal{Z}/\mathcal{F})$$

Consequently we obtain the restricted index class:

$$\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E|\mathcal{Z}) = [E] \cap [\mathcal{D}] \cup [\iota_{M,\mathcal{Z}}] \in K^*(C^*(\mathcal{Z}/\mathcal{F}))$$

Corollary: Let $\mathcal{Z} \subset M$ be a generic closed invariant set containing the support of a holonomy invariant transverse measure Λ. Then there is a well defined measured-index functional

$$\mathrm{Ind}_\Lambda \equiv Tr_\Lambda \circ \mathrm{Ind}_\mathcal{F} \circ \iota^*_{M,\mathcal{Z}} \colon K_*(M) \to \mathbb{R}$$

which factors through $K^*(C^*_r(\mathcal{Z}/\mathcal{F})$.

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We can then consider $\mathcal{Z} \subset M$ as the image of a foliated space $\mathfrak{M}$ and study our problems in this context.

Assumptions:

- $\mathfrak{M}$ is a leafwise orientable matchbox manifold;
- The dynamics of $\mathcal{F}$ on $\mathfrak{M}$ is equicontinuous;
- The leaves of $\mathcal{F}$ are simply connected.

Theorem of Clark & H implies that $\mathfrak{M}$ admits a presentation as an inverse tower of compact manifolds B_ℓ.

That is, given $\epsilon > 0$ there is a compact, orientable base manifold B_{ℓ_0} and a tower of proper orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} \cdots \xrightarrow{p_{\ell_0+2}} B_{\ell_0+1} \xrightarrow{p_{\ell_0+1}} B_{\ell_0}$$

so that $\mathfrak{M}$ is homeomorphic to the inverse limit of this system, and the fibers of the natural map $\pi_{\ell_0} \colon \mathfrak{M} \to B_{\ell_0}$ have diameters $\leq \epsilon$.

The diameter assumption means that the manifold B_{ℓ_0} embeds in M as an "approximate leaf" for $\mathcal{F}$, so that the leaves of $\mathcal{F}|\mathfrak{M}$ are coverings of B_{ℓ_0}. This follows by *shape theory* of $\mathfrak{M} \subset M$.

The assumption that the leaves of $\mathcal{F}|\mathfrak{M}$ are simply connected has a simple interpretation.

Choose basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

Then there is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_{\ell_0+1} \xrightarrow{p_{\ell_0+1}} G_{\ell_0}$$

Set $q_\ell = p_\ell \circ \cdots \circ p_{\ell_0+1} \colon B_\ell \longrightarrow B_{\ell_0}$.

Then the leaves of $\mathcal{F}$ in $\mathfrak{M}$ are simply connected if and only if the *residual kernel* $\displaystyle\bigcap_{\ell \geq \ell_0} (q_\ell)_\#(G_\ell) = \{0\}$.

Geometric interpretation of even index

Suppose that the leaves of $\mathcal{F}$ have even dimension.

Theorem: Let $[E] \in K^0(M)$. Then there is a constant $\lambda_0 > 0$ so that the measured index

$$Tr_\Lambda(\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes E|\mathcal{Z})) = \lim_{\ell \to \infty} \frac{\lambda_0}{d(\ell, \ell_0)} \cdot \mathrm{Ind}(\mathcal{D}_\ell \otimes E|B_\ell)$$

where

- λ_0 depends only on Λ and the choice of B_0
- $d(\ell, \ell_0)$ is the covering degree of $q_\ell \colon B_\ell \longrightarrow B_{\ell_0}$
- $E|B_\ell$ is restriction of $E \to M$ to an embedding of $B_\ell \subset M$
- $\mathcal{D}_\ell$ is a geometric operator on B_ℓ which *approximates* the leafwise operator $\mathcal{D}$.

This result should be compared with the geometric interpretation given to the index for measured laminations by surfaces in:

The ∂-operator, *Appendix A*, *Global analysis on foliated spaces*, by C. C. Moore and C. Schochet, 1988.

In that work, no assumption is made on the dynamics of the lamination, so the above result gives an “improvement” of the conclusions there.

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The assumption that the dynamics of the lamination is *equicontinuous* implies the existence of approximating cycles in M, and these cycles "carry" the measured index.

Geometric interpretation of odd dimension

Suppose that the leaves of $\mathcal{F}$ have odd dimension.

Theorem: Let $\varphi\colon M \to U(N)$ be a leafwise smooth function with values in the unitary group, for some $N > 0$. Then there is a constant $\lambda_0 > 0$ so that the measured index

$$\eta_\Lambda(\mathcal{D}, \varphi) = Tr_\Lambda(\mathrm{Ind}_{\mathcal{F}}(\mathcal{D} \otimes \varphi|\mathcal{Z})) = \lim_{\ell \to \infty} \frac{\lambda_0}{d(\ell, \ell_0)} \cdot \eta(\mathcal{D}_\ell, \varphi)$$

where $\eta_\Lambda(\mathcal{D}, \varphi)$ is the leafwise η-invariant, and $\eta(\mathcal{D}_\ell, \varphi)$ denotes the relative η-invariant for $\mathcal{D}_\ell$ coupled to the twisted flat bundle over B_ℓ defined by the restricted unitary bundle $\varphi|B_\ell$.

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The proof of this result is actually quite technically involved, though may be considered “standard” given the previous works.

This result should be compared with the geometric interpretation given to the odd index for flat bundles in the papers:

Toeplitz operators and the eta invariant: the case of S^1, (with Ronald G. Douglas and Jerome Kaminker), in *Index theory of elliptic operators, foliations, and operator algebras*, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988.

Eta invariants and the odd index theorem for coverings, in *Geometric and topological invariants of elliptic operators*, Contemp. Math., 105, Amer. Math. Soc., Providence, RI, 1990.

Both of these papers implicitly assumed that the dynamics of the foliations being considered were equicontinuous.

Conclusions

The approach to studying index theory for foliations considered previously can be understood as special cases of the study of the index theory for a particular classes of foliated spaces.

Introducing generalized foliation cycles, as represented by matchbox manifolds, gives approaches to generalizing these results.

Conclusions

For example, the techniques can be applied to obtain:

• Relations between spectra for matchbox manifolds and operators from quasi-crystals;

• Measured spectral flow estimates (integrated density of states) for leafwise operators in foliations

• Relation between pure-point spectrum and index theory

• Spectrum for almost $1-1$ factors (generalizing the Denjoy flow)

Conclusions

An important open problem is then:

Problem: Extend the above results for generalized foliation cycles represented by equicontinuous matchbox manifolds, to those represented by minimal expansive matchbox manifolds.

Thank you for your time and patience.

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The slides for both talks can be downloaded from:

http://www.math.uic.edu/∼hurder/talks/Nagoya20120321.pdf

http://www.math.uic.edu/∼hurder/talks/Nagoya20120324.pdf