Dynamical Invariants of Foliations[1]

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What is a foliation?

A foliation $\mathcal{F}$ of codimension-q on a compact manifold is . . .

• a local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle for a "good covering". (Ehresmann, Haefliger)

• a dynamical system on M with multi-dimensional time.

• a groupoid over $\Gamma_{\mathcal{F}} \to M$ with fibers complete manifolds, the holonomy covers of leaves. (Winkelnkemper, Connes)

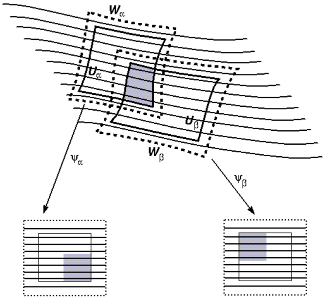
Each point of view has advantages and disadvantages for the study and applications of foliation theory.

Problem: How to distinguish foliations, up to diffeomorphism for example?

Basic definition

Let M be a smooth manifold of dimension n.

Definition: M is a C^r *foliated manifold* if the leafwise transition functions for the foliation charts $\varphi_i\colon U_i \to [-1,1]^n \times T_i$ (where $T_i \subset \mathbb{R}^q$ is open) are C^∞ leafwise, and vary C^r with the transverse parameter in the leafwise C^∞-topology.



Riemannian foliations

$\mathcal{F}$ is a *Riemannian foliation* if there is a Riemannian metric on TM so that its restriction to $Q = TM/T\mathcal{F}$ is invariant under the leafwise parallelism.

Theorem: [Molino] Let $\mathcal{F}$ be a smooth Riemannian foliation on a compact manifold M. Then for each leaf L, its closure $\overline{L}$ is a manifold, and:

1. the restricted foliation $\mathcal{F} \mid \overline{L}$ is Riemannian, even homogeneous;
2. all leaves of $\mathcal{F}$ in $\overline{L}$ are dense in $\overline{L}$;
3. the closures of the leaves form a singular Riemannian foliation of M.

Moreover, if the group of foliated homeomorphisms of M is transitive, then the foliation by leaf closures is defined by a submersion to the quotient manifold $W = M/\overline{T\mathcal{F}}$.

Remark: Molino's Theorem and related works by Carrière, Ghys, and others give (almost) a complete classification of Riemannian foliations in low dimensions.

Anosov foliations

Anosov foliations are at the opposite extreme from Riemannian foliations:

$$TM = E^+ \oplus \langle \vec{X} \rangle \oplus E^-$$

where the flow φ_t of $\vec{X}$ uniformly expands E^+, and uniformly contracts E^-.

$\mathcal{F}^\pm$ is the foliation given by the integral manifolds of the distribution $\vec{X} \oplus E^\pm$.

The restriction of a Riemannian metric on TM to $Q^\pm \cong E^\mp$ is either uniformly contracted/expanded under the leafwise parallelism along $\vec{X}$.

Anosov foliations are extremely well-studied, and though not classified, there are algebraic models for "what they should be".

Concept extends to actions of Lie groups, and suspensions of countable groups acting smoothly on compact manifolds.

Intermediate classes of foliation dynamics

The two cases above have exceptional uniformity in their structure:

• For a Riemannian foliation $\mathcal{F}$, all leaves have diffeomorphic and quasi-isometric holonomy coverings, given by a "typical" leaf $L \subset M$.

• For an Anosov foliation $\mathcal{F}$, all leaves are diffeomorphic and quasi-isometric, to either $\mathbb{R}^n$ or to a simply-connected nil-manifold $\mathcal{N}^n$.

For the general foliation $\mathcal{F}$, the closure of a leaf L can contain a variety of diffeomorphism types of manifolds as leaves (Reeb foliation, and beyond.)

Thus, the "dynamics of $\mathcal{F}$" must be defined "locally", as there is no uniform notion of "time".

Pseudogroups

A section $\mathcal{T} \subset M$ for $\mathcal{F}$ is an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ on $\mathcal{T}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$.

Definition: A pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if there is

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all orbits of $\mathcal{G}$;
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}$ which generates $\mathcal{G}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Groupoid word length

Definition: The groupoid of $\mathcal{G}$ is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \;\&\; x \in D(g)\}\ ,\ \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

The *word length* $\|[g]\|_x$ of the germ $[g]_x$ of g at x is the least k such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_k}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another along an orbit or leaf, while preserving the germinal dynamics.

Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: The *normal derivative cocycle* $D \colon \Gamma_{\mathcal{G}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D([g]_x) = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

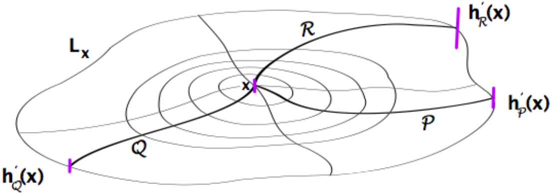
which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D([h]_y) \cdot D([g]_x)$$

Asymptotic exponent

Definition: The *transverse expansion rate* function at x is

$$\lambda(\mathcal{G}, k, x) = \max_{\|[g]\|_x \leq k} \frac{\ln(\|D_x g\|)}{k} \geq 0$$



Definition: The *asymptotic transverse growth rate* at x is

$$\lambda(\mathcal{G}, x) = \limsup_{k \to \infty} \lambda(\mathcal{G}, k, x) \ \geq \ 0$$

This is essentially the "maximum Lyapunov exponent" for $\mathcal{G}$ at x.
$\lambda(\mathcal{G}, x)$ is a Borel function of $x \in \mathcal{T}$, as each norm function $\|D_{w'} h_{\sigma_{w,z}}\|$ is continuous for $w' \in D(h_{\sigma_{w,z}})$ and the maximum of Borel functions is Borel.

Lemma: $\lambda_{\mathcal{F}}(z)$ is constant along leaves of $\mathcal{F}$.

Expansion classification

Theorem: (Hurder, 2000, 2005) Let $\mathcal{F}$ be a C^1-foliation of compact manifold M. Then there is a disjoint decomposition

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

consisting of $\mathcal{F}$-saturated, Borel subsets of M, defined by:

1. Elliptic points: $\mathcal{E} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \forall\ k \geq 0,\ \lambda(\mathcal{G}, k, x) \leq \kappa(x)\}$
   i.e., "points of bounded expansion" (example: Riemannian foliations)
2. Parabolic points: $\mathcal{P} \cap \mathcal{T} = \{x \in \mathcal{T} - (\mathcal{E} \cap \mathcal{T}) \mid \lambda(\mathcal{G}, x) = 0\}$
   i.e., "points of slow-growth expansion" (example: Distal foliations)
3. Partially Hyperbolic points: $\mathcal{H} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}, x) > 0\}$
   i.e., "points of exponential-growth expansion" (example: Anosov foliations, or more generally, non-uniformly, partially hyperbolic foliations)

Classification - Parabolic & Elliptic Case

Example: If $\mathcal{F}$ is a foliation with all leaves compact, then $\lambda(\mathcal{G}, x) = 0 \quad \forall\, x \in \mathcal{T}$.

Theorem: If there exists $\kappa > 0$ so that $\lambda(\mathcal{G}, k, x) \leq \kappa \quad \forall\, x \in \mathcal{T}$, then $\mathcal{F}$ is equicontinuous (and so Riemannian?)

The case where there is a bound $\kappa(x)$ depending on x is a mystery!

The parabolic case $\lambda(\mathcal{G}, x) = 0 \quad \forall\, x \in \mathcal{T}$ will be discussed later.

Godbillon-Vey classes

Assume that $\mathcal{F}$ is a C^2-foliation and the normal bundle $Q = TM/T\mathcal{F}$ is oriented.

Let ω be a q-form defining $T\mathcal{F}$, so then $d\omega = \eta \wedge \omega$ for a 1-form η.

The form $\eta \wedge (d\eta)^q$ is then a closed $2q+1$-form on M, and its cohomology class is independent of choices.

$$GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H^{2q+1}_{deR}(M)$$

is the *Godbillon-Vey class* of $\mathcal{F}$.

Question: (Moussu-Pelletier 1974, Sullivan 1975) If $GV(\mathcal{F}) \neq 0$, what does this imply about the dynamical properties of $\mathcal{F}$? Must there be leaves with exponential growth?

This question motivated many developments in the study of foliations.

Godbillon-Vey and exponent

Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^2-foliation of codimension-q with $GV(\mathcal{F}) \neq 0$. Then the hyperbolic set $\mathcal{H}$ has *positive Lebesgue measure*.

Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^1-foliation of codimension-1 such that the hyperbolic set $\mathcal{H}$ has positive Lebesgue measure. Then $\mathcal{F}$ has a resilient leaf, and hence has positive geometric entropy.

Corollary: (Duminy 1982) Let $\mathcal{F}$ be a C^2-foliation of codimension-1 with $GV(\mathcal{F}) \neq 0$. Then $\mathcal{F}$ has a resilient leaf.

Combining results, we have the stronger statement:
Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^1-foliation of codimension-1 such that the hyperbolic set $\mathcal{H}$ has positive Lebesgue measure. Then $\mathcal{F}$ has positive geometric entropy, in the sense of Ghys-Langevin-Walczak, so that $GV(\mathcal{F}) \neq 0$ implies positive entropy.

Minimal sets

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{W}$

M compact, then minimal sets for foliations always exist.

Transitive sets are most important for flows – Axiom A attractors are transitive sets, while the minimal sets include the periodic orbits in the domain of attraction.

Question: Can you describe the minimal sets for $\mathcal{F}$ in each type of dynamic?

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap \mathcal{H} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

What other sorts of parabolic minimal sets are there?

Proposition: A parabolic minimal set has zero geometric entropy.

Question: What are the zero entropy minimal sets?

Solenoids, and weak solenoids

Weak solenoids are generalizations to higher dimensions of 1-dimensional p-adic solenoids, i.e. inverse limit of finite-to-one coverings of a circle

$$\mathbb{S}_\infty = \lim_{\longleftarrow} \left\{ p_i : \mathbb{S}^1 \to \mathbb{S}^1, \ i \geq 0 \right\} .$$

An *n-dimensional solenoid*, as studied by McCord (1965) and Fokkink and Oversteegen (2002) is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \ \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps.

$\mathcal{S}$ is a fibre bundle with Cantor set fibre, and profinite structure group.

If all defined covering maps $L_{\ell+1} \to L_0$ are normal (Galois) coverings, then $\mathcal{S}$ is called a McCord solenoid, and otherwise is a (weak) solenoid.

Generalized solenoids

Williams (1970, 1974) introduced a broader class of inverse limit spaces, called generalized solenoids, which model Axiom A attractors.

Let K be a branched manifold, i.e. each $x \in K$ has a neighborhood homeomorphic to the disjoint union of a finite number of Euclidean disks modulo some identifications.

$f : K \to K$ is an expansive immersion of branched manifolds satisfying a flattening condition. Then in

$$\mathcal{K} = \varprojlim \{ f : K \to K \} ,$$

each point x has a neighborhood homeomorphic to $[-1, 1]^n \times \text{Cantor set}.$

Universality Properties

For these abstract solenoidal spaces, the basic question is whether they are homeomorphic to minimal sets of parabolic or hyperbolic type?

Embedding Property: Given a (generalized) solenoid $\mathcal{S}$, does there exists a C^r-foliation $\mathcal{F}_M$ of a compact manifold M and an embedding of $\iota\colon \mathcal{S} \hookrightarrow M$ as a foliated subspace? $(r \geq 0)$

Germinal Extension Property: Given a (generalized) solenoid $\mathcal{S}$, does there exists a C^r-foliation $\mathcal{F}_U$ of an open manifold U and an embedding of $\iota\colon \mathcal{S} \hookrightarrow U$ as a foliated subspace? $(r \geq 0)$

Solutions to the embedding problem for solenoids modeled on $\mathbb{S}^1$ were given by Gambaudo, Tressier, et al in 1990's.

For the general cases of weak or generalized solenoids, these questions is related to the *Pisot Conjecture* for tiling spaces, but seems not considered more generally.

Embeddings of total solenoids

Theorem: [Clark-H 2008] Let $\mathcal{F}_0$ be a C^r-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}$ on M which is C^r-close to $\mathcal{F}_0$, and $\mathcal{F}$ has a solenoidal minimal set contained in U with base L_0. If $\mathcal{F}_0$ is a distal foliation, then $\mathcal{F}$ is also distal.

The criteria for embedding depends on the degree of smoothness required, and the tower of subgroups of the fundamental group.

Instability

One application is a type of "Reeb Instability" result:

Theorem: [Clark & H 2010] Let $\mathcal{F}_0$ be a C^∞-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0;\mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some saturated open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}_M$ on M which is C^∞-close to $\mathcal{F}_0$, and $\mathcal{F}_M$ has an uncountable set of solenoidal minimal sets $\{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$, which are *pairwise non-homeomorphic*.

Solenoid-type objects are "typical" for perturbations of dynamical systems and possibly also for foliations, so study them to understand general problems about foliations.

Embedding Williams solenoids

The Denjoy minimal set for a flow on $\mathbb{T}^2$ is the simplest example of a Williams solenoid. It is embedded into a C^1-foliation.

Ronald Knill gave construction of smooth foliated embedding or the Denjoy minimal set, for flows of codimension 2.

Both examples are contained in parabolic sets.

Work In Progress: Find other examples and constructions of foliated embeddings of generalized solenoidal sets in foliated manifolds, and find obstructions to making such embeddings