Solenoidal minimal sets for smooth dynamics

Steve Hurder
University of Illinois at Chicago

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Systémes Dynamiques et Systémes Complexes

Une conférence pour célébrer les 60 ans de Jean-Marc Gambaudo

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A trip back in time...

A quote by Jean-Marc, from his 1992 paper with Charles Tresser *"Self Similar Constructions in Smooth Dynamics: Rigidity, Smoothness and Dimension"*:

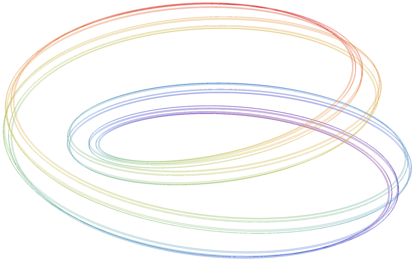
> *There are relations between the optimal smoothness of some examples in dynamics and dimension*

The study of the regularity of invariant structures for smooth dynamical systems has long been an active topic:

- Anosov observed in 1967 that the stable and unstable foliations for a smooth Anosov diffeomorphism have $C^{1+\alpha}$ smoothness;
- Hirsch and Pugh in 1968 showed that stable manifolds for hyperbolic sets have $C^{1+\alpha}$ smoothness;
- Schweitzer in 1971 produced $C^{1+\alpha}$ counterexamples to the Seifert Conjecture for flows on compact 3-manifolds;
- Harrison in 1975 showed the existence, for any $r \geq 0$, of C^r-diffeomorphisms of surfaces which are not topologically conjugate to any C^{r+1}-diffeomorphism;

and remains an active topic of research today.

In this talk, we explore a theme that began with Smale's observation, in his celebrated survey article (Bulletin A.M.S. 1967), that the doubling solenoid arises naturally as a non-manifold basic set of a smooth expanding endomorphism of a compact 3-manifold, hence is realized as an invariant set for its expanding foliation, which is a $C^{1+\alpha}$-flow.



The Smale attractor is an example of a classical *Vietoris solenoid*.

Let $\mathbf{P} = (p_1, p_2, \ldots)$ be an infinite sequence of integers, $p_i > 1$.

Let $f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1$ be a p_i-to-1 self-covering map of a circle.

A Vietoris solenoid is the inverse limit space

$$\Sigma_{\mathbf{P}} = \{(y_i) = (y_0, y_1, y_2, \ldots) \mid f^i_{i-1}(y_i) = y_{i-1} \ \} \subset \prod_{i \geq 0} \mathbb{S}^1$$

with subspace topology from the Tychonoff topology on $\prod_{i \geq 0} \mathbb{S}^1$.

There is a projection map $\Pi \colon \Sigma_{\mathbf{P}} \to \mathbb{S}^1$, $\Pi(y_0, y_1, y_2, \ldots) = y_0$.

The fibre $\mathfrak{X}_b = \Pi^{-1}(b) = \{(b, y_1, y_2, \ldots)\} \subset \Sigma_{\mathbf{P}}$ is a Cantor set, for each $b \in \mathbb{S}^1$. For an embedded solenoid, $\mathfrak{X}_b$ corresponds to the intersection of the solenoid with a transverse section.

The fundamental group $\pi_1(\mathbb{S}^1, b) = \mathbb{Z}$ acts on $\mathfrak{X}_b$ via lifts of paths in $\mathbb{S}^1$, so the monodromy action on the fiber defines a group action $\Phi \colon \mathbb{Z} \times \mathfrak{X}_b \to \mathfrak{X}_b$ which is a classical odometer action.

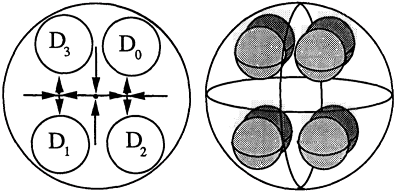
The *meta principle* is to study relations between properties of the monodromy action, and the dynamics of extensions of the action to a diffeomorphism of a manifold.

For example, Gambaudo, van Strien and Tresser, in their work "*The periodic orbit structure of orientation-preserving diffeomorphisms on $\mathbb{D}^2$ with topological entropy zero*", Annales de l'I.H.P. (1989), showed that the periodic orbits for an entropy zero, orientation-preserving C^1 diffeomorphism $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ have a "tree structure": They summarized their results as saying

> *All periodic orbits of such an orientation-preserving diffeomorphism of $\mathbb{D}^2$ can be organized in a tree: each orbit of period $n \geq 2$ has a parent orbit.*

This "tree structure" on periodic orbits persists through the various generalizations we discuss here.

Gambaudo and Tresser showed in another work, "*Self-similar constructions in smooth dynamics: rigidity, smoothness and dimension*", C.M.P., 1992, the existence of a C^k-diffeomorphism on $\mathbb{D}^d$, where k increases with $d \geq 2$, so that the periodic orbits of the map satisfy a period doubling property.



The problem we consider in our work dates from 1965, when McCord began the study of higher-dimensional generalizations of the Vietoris solenoid.

Let M_0 be a connected closed manifold, and let $f^i_{i-1} : M_i \to M_{i-1}$ be a sequence of finite-to-one proper covering maps. Then

$$\mathfrak{M}_\infty = \varprojlim\{f^i_{i-1} : M_i \to M_{i-1} \mid i \geq 1\}$$

is a compact connected metrizable space, called a *(weak) solenoid*.

There is a fibration map $\Pi_0 \colon \mathfrak{M}_\infty \to M_0$, and for $b \in M_0$ the fiber $\mathfrak{X}_b = \Pi_0^{-1}(b)$ is a Cantor space. The monodromy of the fibration yields a group action $\varphi\colon G \times \mathfrak{X}_b \to \mathfrak{X}_b$ where $G = \pi_1(M_0, b)$.

Theorem [McCord, 1965]. A solenoid is a *matchbox manifold*, or *generalized lamination* (Ghys), or *solenoidal manifold* (Sullivan).

Question: Let $\mathfrak{M}_\infty$ be a weak solenoid. For what $1 \leq k \leq \omega$, does there exists a transversally C^k-foliation $\mathcal{F}$ such that $\mathfrak{M}_\infty$ is *homeomorphic* to a minimal set of $\mathcal{F}$.

This natural question is not so easy to answer.

In a paper with Alex Clark, "*Embedding solenoids in foliations*", Top. Apps. 2011, we used a generalization of the Tresser and Gambaudo construction to give an answer for toroidal solenoids.

Theorem (Clark-H, 2011): Let $\mathfrak{M}_\infty$ be a solenoid defined by a tower of coverings of the n-torus $\mathbb{T}^n$, then there exist conditions on $1 \leq k \leq \omega$ and $d \geq 2$, which imply that there exists a codimension-d, transversally C^k-foliation $\mathcal{F}$ such that $\mathfrak{M}_\infty$ is homeomorphic to a minimal set of $\mathcal{F}$.

These conditions are given in Propositions 8.5 and 8.6 in our paper. They are more subtle than the estimates in the 1992 work of Gambaudo & Tresser – the embedding criteria depend on a *presentation* of the solenoid.

In contrast to the "elementary" toroidal solenoids, which are compact abelian groups, Schori constructed in his 1966 paper, a 2-dimensional solenoid which is not homogeneous as a topological space. It is defined via an tower of coverings of a genus-2 surface.

In a paper "*Molino theory for matchbox manifolds*", Pacific Journal Math, 2017, with Jessica Dyer and Olga Lukina, we show:

Theorem: The mondromy action of the Schori solenoid is not LQA, hence the solenoid does not embed into a real analytic foliation of any codimension.

In this talk, we will take a tour through the techniques for the study of weak solenoids, which results in a proof of conclusion.

The first stop on our tour, we consider the structure of the monodromy actions on the fibers of weak solenoids.

Let $\mathfrak{X}$ be a Cantor set with metric $d_{\mathfrak{X}}$ and let G be a group.

Definition: An action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ by homeomorphisms is equicontinuous if for every $\epsilon > 0$ there exist $\delta > 0$ such that

$$d_{\mathfrak{X}}(x, y) < \delta \implies d_{\mathfrak{X}}(\varphi(g)(x), \varphi(g)(y)) < \epsilon \quad \text{for all } g \in G.$$

Fact: The monodromy action of a weak solenoid is equicontinuous.

Question: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an *equicontinuous minimal* Cantor action. Does there exists a C^k-action $\Phi\colon G \times \mathbb{D}^d \to \mathbb{D}^d$ which has an invariant minimal set $K \subset \mathbb{D}^k$, such that the restricted action $\Phi|K$ is conjugate to the action of φ on $\mathfrak{X}$?

We recall some classical notions from Auslander, **Minimal flows and their extensions**, 1988.

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ if U is a non-empty clopen subset, and for any $g \in G$, $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. That is, the translates of U form a partition of the Cantor set $\mathfrak{X}$. It follows that

$$G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\} \tag{1}$$

is a subgroup of finite index in G, called the *stabilizer* of U.

For a Cantor space $\mathfrak{X}$, let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (compact open) subsets of $\mathfrak{X}$. Note that for $\phi \in \mathbf{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$.

Proposition (Glasner and Weiss, 1995): A Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if and only if, for the induced action $\Phi_* \colon G \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$, the G-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite.

Corollary: Let $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous Cantor action. Then for all $x \in \mathfrak{X}$ and all $\delta > 0$, there exists an adapted clopen set U with $x \in U \subset B_{\mathfrak{X}}(x, \delta)$.

Definition: Let $\varphi_i \colon G_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be minimal equicontinuous Cantor actions, for $i = 1, 2$. Say that φ_1 is return equivalent to φ_2 if there exist

- adapted clopen subsets $U_i \subset \mathfrak{X}_i$ for $i = 1, 2$
- a homeomorphism $h \colon U_1 \to U_2$

such that h induces an isomorphism $\alpha_h \colon G_1|U_1 \to G_2|U_2$ of the restricted groups, where $G_i|U_i \subset \mathbf{Homeo}(U_i)$.

Remark: When $U_i = \mathfrak{X}_i$ for $i = 1, 2$, and the actions are effective, this reduces to the notion of topological conjugacy of the actions, where $\alpha_h \colon G_1 \to G_2$ intertwines the actions.

Theorem (Clark, H, Lukina 2017): Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be weak solenoids. Suppose that there exists a homeomorphism $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$, then the fiber monodromy actions associated to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.

There is a converse to this result, but it requires assumptions on the base manifold and possibly further assumptions on the actions. Here is a sample result:

Theorem (Clark, H, Lukina 2017): Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are toroidal solenoids of the same dimension n. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if the fiber monodromy actions associated to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.

Problem: Find invariants of the return equivalence class of an action which either are sufficient to imply the action admits a smooth realization, or which imply that no such realization exists.

The results of Gambaudo and Tresser referred to above, specify that the periodic orbits of the extended smooth $\mathbb{Z}$-action satisfy a period doubling property. On the other hand, the embedding problem as formulated above need not have this property. For example, the presentation of the doubling solenoid over $\mathbb{S}^1$ can be defined by covering maps with degrees 2^n where n tends rapidly to infinity, so the embedding obtained may only have periodic orbits that grow in order by these covering degrees.

In order to explore the aspects of the embedding question, we introduce the group chain (or odometer) model for an equicontinuous minimal Cantor action.

Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous Cantor action.

For a choice of basepoint $x \in \mathfrak{X}$ and scale $\epsilon > 0$, there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \epsilon$. Iterating this construction, for a given basepoint x, one can always construct the following:

Definition: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous action on a Cantor space $\mathfrak{X}$. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is said to be an *adapted neighborhood basis* at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap\ U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.

For such a collection, setting $G_\ell = G_{U_\ell}$ we obtain a descending chain of finite index subgroups

$$\mathcal{G}_{\mathcal{U}} = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\} \; .$$

Set $X_\ell = G/G_\ell$ and note that G acts transitively on the left on X_ℓ. The inclusion $G_{\ell+1} \subset G_\ell$ induces a natural G-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$\mathfrak{X}_\infty \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell > 0\}$$

which is a Cantor space with the Tychonoff topology, and the action on the factors X_ℓ induces a minimal equicontinuous action $\Phi_x \colon G \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$.

The action $\Phi_x \colon G \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$ is called the *generalized odometer* model, or also called a *subodometer*, by Cortez & Petite in their work "*G-odometers and their almost one-to-one extensions*", 2008.

We give some remarks on this construction.

- Each $X_i = G/G_i$ is a finite set with a left action of G. It is a group if G_i is normal in G, and then the Cantor space $\mathfrak{X}_\infty$ is a profinite group.

- The intersection $K(\mathcal{G}_\mathcal{U}) = \bigcap_{\ell \geq 0} G_\ell$ is called the kernel of $\mathcal{G}_\mathcal{U}$.

- For $g \in K(\mathcal{G}_\mathcal{U})$, the left action of g on X_ℓ fixes the coset $e_\ell \in X_\ell$ and hence fixes the limiting point $e_\infty \in \mathfrak{X}_\infty$.

We next recall the tree model for the action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$.

The construction is analogous to that used by Gambaudo in his 1992 paper with Tresser, and also used in his work with Martens, "*Algebraic topology for minimal Cantor sets*", Ann. Henri Poincaré 2006.

First, choose an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action, $\mathcal{U} = \{U_i \subset \mathfrak{X} \mid i \geq 1\}$.

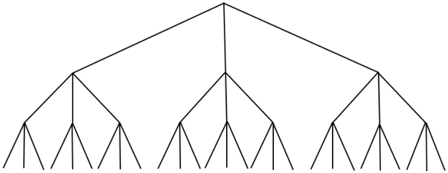
Note that by assumption we have $\bigcap U_i = \{x\}$.

Next, associate a vertex $v_{i,g}$ at level i to each $g \cdot U_i$.

Join $v_{i,g}$ and $v_{i+1,h}$ by an edge if and only if $h \cdot U_{i+1} \subset g \cdot U_i$.

A sequence of vertices $(v_{i,g_i})_{i \geq 0}$ is a path in the space $\mathcal{P}_T$ of paths in T, and $\mathfrak{X} \cong \mathcal{P}_T$.

The subgroup G_i of elements which stabilize U_i has finite index in G, and there is a group chain of stabilizers $\mathcal{G}_{\mathcal{U}} \equiv \{G_i\}_{i \geq 0}$ associated to the action.



We next introduce the Ellis group associated to an action.

An action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ induces a representation $\Phi\colon G \to \mathbf{Homeo}(\mathfrak{X})$ with image group

$$H_\Phi = \Phi(G) \subset \mathbf{Homeo}(\mathfrak{X})$$

Definition: The closure $E(\Phi)$ of H_Φ, in the topology of pointwise convergence on maps, is called the *Ellis (enveloping) semigroup*.

Proposition (Ellis, 1969): Let φ be an equicontinuous Cantor action. Then $E(\Phi) = \overline{H_\Phi}$ = closure of H_Φ in the *uniform topology on maps*. In particular, $\overline{H_\Phi}$ is a profinite group.

For $x \in \mathfrak{X}$ let $\overline{H_\Phi}_x = \{h \in \overline{H_\Phi} \mid h(x) = x\}$ be its isotropy group.

Lemma: The left action of $\overline{H_\Phi}$ on $\mathfrak{X}$ is transitive, hence $\mathfrak{X} \cong \overline{H_\Phi}/\overline{H_\Phi}_x$ and the closed subgroup $\overline{H_\Phi}_x \subset \overline{H_\Phi}$ is independent of the choice of basepoint x, up to topological isomorphism.

We give a representation for $\overline{H_\Phi}_x$ in terms of the odometer model for the action.

The <u>normal core</u> N of a subgroup $H \subset G$ is the largest subgroup $N \subset H$ which is normal in G.

Let $C_i \subset G_i$ be the normal core of G_i in G, then C_i has finite index in G. Define the profinite group

$$G_\infty \equiv \varprojlim \; \{q_i \colon G/C_{i+1} \to G/C_i \mid i > 0\} \; .$$

Each group G/C_i acts on the finite set $X_i = G/G_i$, so there is an induced action $\widehat{\Phi}_\infty \colon G_\infty \to \mathbf{Homeo}(\mathfrak{X}_\infty) \cong \mathbf{Homeo}(\mathfrak{X})$.

Theorem (Dyer-H-Lukina, 2016). $\overline{H_\Phi} \cong \widehat{\Phi}_\infty(G_\infty)$, and

$$\mathcal{D}_\infty \equiv \varprojlim \; \{\pi_i \colon G_{i+1}/C_{i+1} \to G_i/C_i \mid \ell \geq 0\} \;\cong \overline{H_{\Phi_x}} \; . \qquad (2)$$

The inverse limit group $\mathcal{D}_\infty$ is called the *discriminant group* for the action. Its non-triviality is the obstruction to the existence of a transitive right action on $\mathfrak{X}$ that commutes with the left action φ.

We next return to considering the structure of weak solenoids.

The approach to the study of weak solenoids via their monodromy Cantor actions obtained from group chains was initiated in the work of [Fokkink & Oversteegen, 2002].

Let $\Pi_0 \colon \mathfrak{M}_\infty \to M_0$ be a weak solenoid defined by the system of maps $\{f_0^i \colon M_i \to M_0 \mid i > 0\}$, where $f_0^i = f_0^1 \circ \cdots \circ f_{i-1}^i$.
Choose a basepoint $b \in M_0$ and basepoints $x_i \in M_i$ such that $f_0^i(x_i) = b$. Set $x = \lim x_i \in \mathfrak{X}_b \equiv \Pi_0^{-1}(b)$.

Define $G = G_0 = \pi_1(M_0, b)$, and let $G_i \subset G$ be the subgroup defined by $G_i = \mathrm{Image}\{(f_0^i)_\# \colon \pi_1(M_i, x_i) \to \pi_1(M_0, b)\}$.

$\{G_i \mid i \geq 0\}$ is a descending chain of subgroups of finite index in G.

The subgroups G_i are not assumed to be normal in G.

Example 1: Consider the Vietoris solenoid

$$\Sigma = \{f^i_{i-1} : \mathbb{S}^1 \to \mathbb{S}^1\}.$$

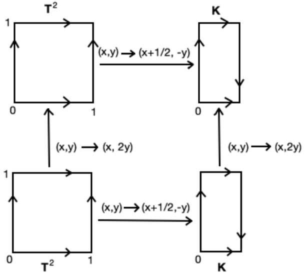
Then $G = \pi_1(\mathbb{Z}, 0) = \mathbb{Z}$, and $G_i = (p_1 \cdots p_i)\,\mathbb{Z}$, where p_i is the degree of f^i_{i-1}.

Then $G/G_i = \mathbb{Z}/p_1 \cdots p_i\mathbb{Z}$.

Since $\mathbb{Z}$ is abelian, $G_i = C_i$, and so G_i/C_i is a trivial group.

Thus $C_\infty \cong \mathfrak{X}_b$, where $\mathfrak{X}_b$ is a fibre of $\Sigma \to \mathbb{S}^1$, and so the discriminant group $\mathcal{D}_\infty$ of the Vietoris solenoid is trivial.

Example 2: Here is a more interesting example, with $\mathcal{D}_\infty$ non-trivial. It is due to [Rogers & Tollefson, 1971/72].



Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and consider an involution

$$r \times i(x,y) = (x + \frac{1}{2}, -y).$$

The quotient $K = \mathbb{T}^2/(x,y) \sim r \times i(x,y)$ is the Klein bottle.

The double cover $L \colon \mathbb{T}^2 \to \mathbb{T}^2 \colon (x,y) \mapsto (x, 2y)$ induces a double cover $p : K \to K$.

Define K_∞ to be the inverse limit of the iterations of $p : K \to K$.

Since $i \circ L = p \circ i$, there is a double cover $i_\infty : \mathbb{T}_\infty \to K_\infty$.

The fundamental group of the Klein bottle is

$$G_0 = \pi_1(K, 0) = \langle a, b \mid bab^{-1} = a^{-1} \rangle.$$

For the cover $p : K \to K$ we have

$$p_* \pi_1(K, 0) = \langle a^2, b \mid bab^{-1} = a^{-1} \rangle,$$

and for $p^n = p \circ \cdots \circ p : K \to K$ we have

$$G_n = (p^n)_* \pi_1(K, 0) = \langle a^{2^n}, b \mid bab^{-1} = a^{-1} \rangle.$$

The cosets of G/G_n are represented by $a^i G_i$, $i = 0, \ldots, n-1$,

$$C_n = \bigcap_{g \in G} g G_n g^{-1} = \langle a^{2^n} \mid bab^{-1} = a^{-1} \rangle.$$

Then $G_n / C_n = \{C_n, bC_n\}$, and so $\mathcal{D}_\infty \cong \mathbb{Z}/2\mathbb{Z}$.

The discriminant need not be an invariant of return equivalence for an equicontinuous Cantor action! We use the tree model for the action to analyze this.

Recall that $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action, and $\mathcal{P}_T$ denotes the space of infinite paths starting at the root point corresponding to $\mathfrak{X}$.

Then we have a minimal action $\Phi\colon G \times \mathcal{P}_T \to \mathcal{P}_T$.

The group G_i stabilizes a branch of a tree, i.e fixes a vertex at level i. Then by minimality of the action, the set of vertices at level i is identified with G/G_i. There is a homeomorphism

$$\phi : \mathcal{P}_T \to \mathfrak{X}_\infty = \varprojlim \{G/G_i \to G/G_{i-1}\},$$

equivariant with respect to the actions of G on $\mathcal{P}_T$ and $\mathfrak{X}_\infty$.

The core subgroup $C_i = \bigcap_{g \in G} gG_ig^{-1} \subset G$ fixes every vertex at level i, and the quotient group G/C_i acts transitively on the set of vertices at level i, which correspond to the set G/G_i. Then

$$C_\infty = \varprojlim \{G/C_i \to G/C_{i-1}\}$$

is a profinite group acting transitively on the path space $\mathcal{P}_T$.

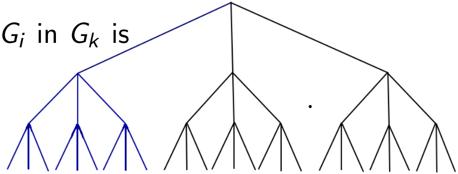
We use this model to consider the discriminant groups of the action φ restricted to an adapted clopen subset $U \subset \mathfrak{X}$.

The set of paths through vertex v_k at level k is a clopen set $U_k \subset \mathcal{P}_T$. Assume that $x \in U_k$.

The restricted action on U_k is given by $\Phi_k \colon G_k \to Homeo(U_k)$.

For each $k \geq i$, the normal core of G_i in G_k is

$$C_{k,i} = \bigcap_{g \in G_k} gG_ig.$$



Observe that $C_{k,i} \supset C_i$ as the action of C_i fixes all vertices at level i, while $C_{k,i}$ fixes just those vertices at level i in the branches of the tree through the vertex v_k.

The isotropy group of the action of $\overline{\Phi(G_k)}$ at x is represented by

$$\mathcal{D}_{x,k} = \varprojlim \{G_i/C_{k,i} \to G_{i-1}/C_{k,i-1} \mid i \geq k\}$$

which is the *discriminant group* of the action $\Phi_k \colon G_k \times U_k \to U_k$.

Note that there are coset inclusions $G_i/C_i \to G_i/C_{k,i}$.

Theorem (Dyer-H-Lukina, 2017) Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action with group chain $\{G_i\}_{i\geq 0}$ associated to a basepoint $x \in \mathfrak{X}$. Then for any $k > j \geq 0$ there is a well-defined surjective homomorphism

$$\Lambda_{k,j} : \mathcal{D}_{x,j} \to \mathcal{D}_{x,k}$$

of discriminant groups.

Definition: The action φ is said to be *stable*, if there exists j_0, such that for all $k > j \geq j_0$ the homomorphism $\Lambda_{k,j}$ is an isomorphism. If no such j_0 exist, then the action is said to be *wild*.

Definition (H-Lukina, 2017): The *asymptotic discriminant* of the action $(\mathfrak{X}, G, \Phi)$ is the equivalence class of the chain of surjective group homomorphisms

$$\mathcal{D}_{x,0} \to \mathcal{D}_{x,1} \to \mathcal{D}_{x,2} \to \cdots$$

with respect to the tail equivalence relation.

The notion of "tail equivalence" is precisely defined in the work with Lukina, *Wild solenoids*, **Transactions A.M.S.**, 2018.

In that work we also show the following key property:

Theorem (H-Lukina, 2017): The asymptotic discriminant of an equicontinuous minimal Cantor action $\varphi \colon G \times \mathfrak{X} \to \mathfrak{X}$ is invariant under the return equivalence of actions. In particular, the property of being stable or wild is an invariant of return equivalence.

Theorem (H-Lukina, 2017). Suppose that $\Phi\colon G \times \mathfrak{X}_\infty \to \mathfrak{X}_\infty$ is a wild action. Then there exists a strictly increasing chain of indices $\{1 \leq k_1 < k_2 < \cdots\}$ so that the sequence of finite subgroups in the profinite group G_∞

$$C_{k_1,\infty} \subset C_{k_2,\infty} \subset \cdots \subset C_{k_j,\infty} \subset \cdots \subset \mathcal{D}_\infty$$

is strictly increasing, where $C_{k_i} = \ker D_{x,\infty} \to D_{x,k_i}$.

We next discuss the notion of an "analytic Cantor action", which was introduced in the works of Alvarez Lopez, and its relation to wildness and the Hausdorff property for the action.

Let $U, V \subset \mathfrak{X}$ be clopen subsets of a Cantor space $\mathfrak{X}$.

• A homeomorphism $h \colon U \to V$ is quasi-analytic (QA) if either $U = V$ and h is the identity map, or for every *clopen* subset $W \subset U$ the fixed-point set of the restriction $h|W \colon W \to h(W) \subset V$ has no interior.

• A homeomorphism $h \colon U \to V$ is locally quasi-analytic (LQA) if for each $x \in U$ there exists a clopen neighborhood $x \in U' \subset U$ such that the restriction $h_{U'} \colon U' \to V' = H(U')$ is QA.

• A group action $\varphi \colon G \colon \mathfrak{X} \to \mathfrak{X}$ is LQA if for each $x \in \mathfrak{X}$, there exists a clopen neighborhood $x \in U$, such that the restrictions of elements of G to U are quasi-analytic.

Remarks:

- A free action $G \times \mathfrak{X} \to \mathfrak{X}$ is quasi-analytic.

- The automorphism group of a spherically homogeneous rooted tree T_d, acting on the Cantor set of ends, is not LQA.

Proposition: Suppose that $\varphi \colon G \colon \mathfrak{X} \to \mathfrak{X}$ is the restriction of a C^ω action on $\mathbb{D}^k$ for some $k \geq 1$. Then the action of φ is LQA.

Theorem (H-Lukina, 2017). Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal Cantor action, where G is finitely generated. Then the action φ is stable if and only if the action of the profinite group G_∞ on $\mathfrak{X}_\infty$ satisfies the LQA property.

These results yield a "non-realizable" criteria:

Corollary. If an equicontinuous minimal Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is not LQA, then any weak solenoid whose mondromy action is return equivalent to this action cannot be realized as the minimal set for a C^ω-foliation.

Question: Is there a version of this result for C^2-foliations?

Corollary: A weak solenoid whose monodromy action is not LQA admits an infinitely increasing chain of closed groups in the fundamental group $\pi_1(M_0, b_0)$ of the base manifold M_0.

Proposition: Let $\mathfrak{M}_\infty$ be a weak solenoid whose base manifold M_0 has nilpotent fundamental group G_0. Then the monodromy action of the solenoid is stable.

We conclude this discussion with a "geometric proof" that an LQA Cantor action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is stable using the tree model for it.

Consider the restricted action of G_k on $U_k \subset \mathfrak{X}$ with group chain $\{G_i\}_{i \geq k}$.

The elements in $C_{k,i} \subset G_i$ stabilize all vertices at level k in a branch of T, while the elements in $C_i \subset C_{k,i}$ stabilize all vertices at level k. Then let

$$S_k = \varprojlim \{C_{k,i}/C_i \to C_{k,i-1}/C_i\} \cong \ker\{\mathcal{D}_x \to \mathcal{D}_{x,k}\}.$$

Suppose that $h \in S_k$, with $h \neq id$, then h acts trivially on U_k, but acts non-trivially on $\mathfrak{X}$. If the action φ is not LQA, then such an h exists for clopen sets U_k with arbitrarily small diameter, and hence the action is not stable.

Finally, we note that the LQA property for a group action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ can be related to properties of the germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \varphi)$ associated to the action.

Recall that for $g_1, g_2 \in G$, we say that $\varphi(g_1)$ and $\varphi(g_2)$ are *germinally equivalent* at $x \in \mathfrak{X}$ if $\varphi(g_1)(x) = \varphi(g_2)(x)$, and there exists an open neighborhood $x \in U \subset \mathfrak{X}$ such that the restrictions agree, $\varphi(g_1)|U = \varphi(g_2)|U$. We then write $\varphi(g_1) \sim_x \varphi(g_2)$.

For $g \in G$, denote the equivalence class of $\varphi(g)$ at x by $[g]_x$. The collection of germs $\mathcal{G}(\mathfrak{X}, G, \varphi) = \{[g]_x \mid g \in G \ , \ x \in \mathfrak{X}\}$ is given the sheaf topology, and forms an *étale groupoid* modeled on $\mathfrak{X}$.

Theorem (H-Lukina, 2017). If an action $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is locally quasi-analytic, then $\mathcal{G}(\mathfrak{X}, G, \varphi)$ is Hausdorff.

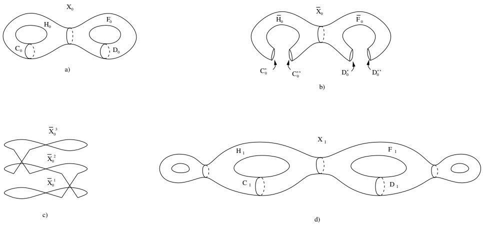
The Hausdorff property for a germinal groupoid $\mathcal{G}(\mathfrak{X}, G, \varphi)$ appears in the work of [Renault, 2008] on the C^*-algebra associated to the action, and has been studied in various works in C^*-algebras.

Problem: Find relations between the wild property for a group action, and the algebraic and topological invariants for the C^*-algebra associated to the action.

Some results on this problem are given in the work by Rui Excel, "*Non-Hausdorff étale groupoids*", **Proc. A.M.S.**, 2011.

We conclude with examples of Cantor actions which are not LQA.

Example 3: [Schori, 1966] gave the first example of a non-homogeneous weak solenoid. It is obtained by taking repeated 3-fold coverings starting with a closed surface Σ_2 of genus 2.



Proposition (Dyer-H-Lukina, 2017). The monodromy action of $G = \pi_1(\Sigma_2, b_0)$ on the fiber of the solenoid over Σ_2 is not LQA, and in particular is wild.

Example 4: Arboreal actions of Galois groups.

The analogy between theory of finite coverings and Galois theory of finite field extensions suggests looking for examples of minimal Cantor actions arising from purely arithmetic constructions.

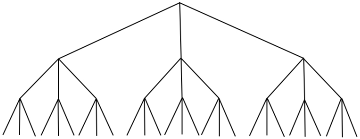
• [R.W.K. Odoni, 1985] began the study of arboreal representations of absolute Galois groups on the rooted trees formed by the solutions of iterated polynomial equations.

• [Jones, 2013] gives a nice introduction and survey of this program, from the point of view of arithmetic dynamical systems and number theory.

The following discussion concerns results of [Lukina, 2018].

Let $X = \mathcal{P}_d$ be the space of paths
in a spherically homogeneous rooted tree T_d.

Let G be any discrete group, acting on T_d
by permuting edges at each level
so that the paths are preserved.
The space of paths with the
cylinder topology is a Cantor set

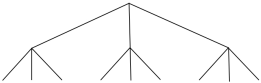


This action is equicontinuous.

Let $f(x)$ be an irreducible polynomial of degree d over a number field K. Let $\alpha \in K$, and suppose $f(x) = \alpha$ has d distinct solutions.

Identify α with the root of a d-ary tree T_d, and identify every solution $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1d}$ of $f(x) = \alpha$ with a vertex at level 1 in the tree.

$\mathrm{Gal}(K(f^{-1}(\alpha))/K)$ is identified with a subgroup of the symmetric group S_d.



For every α_{1i}, consider the equation

$$f(x) = \alpha_{1i}, \text{ so } f \circ f(x) = f(\alpha_{1i}) = \alpha.$$

Suppose there are d^2 distinct roots. Identify the solutions of $f(x) = \alpha_{1i}$ with the d vertices at level 2 connected with α_{1i} at level 1.

The action of $\mathrm{Gal}(K(f^{-2}(\alpha))/K)$ preserves the structure of the tree, so

$$\mathrm{Gal}(K(f^{-2}(\alpha))/K) \subseteq [S_d]^2,$$

where $[S_d]^2$ denotes the two-fold wreath product of symmetric groups S_d.

Continue by induction, assuming that for each $i > 0$ the polynomial $f^i(x)$ has d^i distinct roots.

In the limit, we get a d-ary infinite tree T_d of preimages of α under the iterations of $f(x)$, and the profinite group

$$\mathrm{Gal}_\infty(f) = \varprojlim \{\mathrm{Gal}(K(f^{-i}(\alpha))/K) \to \mathrm{Gal}(K(f^{-(i-1)}(\alpha))/K)\},$$

a subgroup of the infinite wreath product $Aut(T_d) = [S_d]^\infty$.

The group $\mathrm{Gal}_\infty(f)$ is called an *arboreal representation* of the absolute Galois group $\mathrm{Gal}(K^{sep}/K)$.

The representation depends on the polynomial f and on α.

Thus $\mathrm{Gal}_\infty(f)$ is a *profinite* group acting on the Cantor set of paths in the tree T_d.

Example [Odoni, 1985]. If $K = \mathbb{Q}$, $\alpha = 2$, $f(x) = x^2 - x + 1$, then

$$Gal_\infty(f) \cong Aut(T_2) \cong [S_2]^\infty \ .$$

Theorem [Lukina, 2018]. Let $f(x)$ be a polynomial of degree $d \geq 2$ over a field K, suppose all roots of $f^i(x)$ are distinct and $f^i(x) - \alpha$ is irreducible for all $i \geq 0$.

Let $\mathbf{v}$ be a path in the space of paths $\mathcal{P}_d$ of the tree T_d.

Then there exists a countably generated group G_0, a homomorphism $\Phi : G_0 \to Homeo(\mathcal{P}_d)$ and a chain $\{G_i\}_{i\geq 0}$ of subgroups in G_0 such that

(1) There is an isomorphism $\widetilde{\phi} : \overline{\Phi(G_0)} \to \mathrm{Gal}_\infty(f)$,

(2) There is a homeomorphism $\phi : \varprojlim \{G_0/G_i\} \to \mathcal{P}_d$ with $\phi(eG_i) = \mathbf{v}$,

(3) For all $\mathbf{u} \in \mathcal{P}_d$ and $\mathbf{g} \in \overline{\Phi(G_0)}$ we have

$$\widetilde{\phi}(\mathbf{g}) \cdot \phi(\mathbf{u}) = \phi(\mathbf{g}(\mathbf{u})).$$

Theorem [Lukina, 2018]. Suppose the image of an arboreal representation $\mathrm{Gal}_\infty(f)$ is a subgroup of finite index in $Aut(T_d)$. Then the action of the dense subgroup G_0 on the path space $\mathcal{P}_d$ is not LQA, and in particular is wild.

Remark: The proof of this result is geometric, it uses the fact that the action of $Aut(T_d)$ is not locally quasi-analytic.

Remark: There are many techniques, in the literature and developing, for calculating arboreal representations.

Into the future...

Problem: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a wild action. Show that it is not return equivalent to any C^k-action for $k \geq 2$.

Question: Let $\varphi\colon G \times \mathfrak{X} \to \mathfrak{X}$ be a minimal equicontinuous action, where G is a finitely generated, torsion free nilpotent group. Can the action be realized up to return equivalence by a C^k-action for some $k \geq 1$?

Question: Characterize the algebraic number fields and polynomials whose arboreal representations are wild.

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Happy
Birthday