The dynamics of Kuperberg flows

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Introduction

Theorem: [K. Kuperberg] Let M be a closed, oriented 3-manifold. Then M admits a non-vanishing smooth vector field $\mathcal{K}$ without periodic orbits.

• K. Kuperberg, "A smooth counterexample to the Seifert conjecture", Annals of Mathematics, 1994.
• S. Matsumoto, "Kuperberg's C^∞ counterexample to the Seifert conjecture", Sūgaku, Mathematical Society of Japan, 1995.
• É. Ghys, "Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)", Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, 1995.

Theorem: [Ghys, Matsumoto, 1994] A Kuperberg flow Φ_t has a *unique minimal set* Σ.

Problem: *Describe the topological shape of* Σ*, and analyze the dynamics of* Φ_t *restricted to open neighborhoods of* Σ*.*

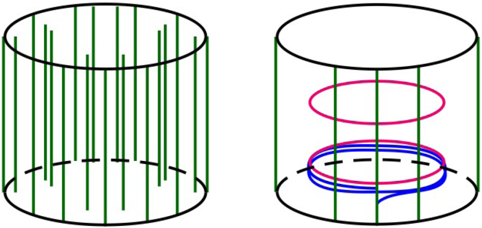
Theorem: [Katok, 1980] If a smooth flow on M^3 has positive topological entropy, then it has periodic orbits.

Hence, the Kuperberg flow Φ_t has topological entropy 0.

Problem: *What type of entropy-zero dynamical system does the restricted flow* $\Phi_t|\Sigma$ *flow yield? For example, is it an odometer? Does its type depend on the construction of the flow?*

Plugs

A plug $\mathbb{P} \subset \mathbb{R}^3$ is a 3-manifold with boundary, with a non-vanishing vector field that agrees with the vertical field on the boundary of $\mathbb{P}$:

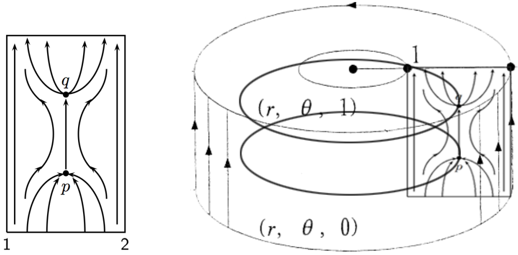


Mirror Symmetry Property: An orbit entering a plug (from the bottom) either never leaves the plug (it is "trapped"), or exits the plug at the mirror image point at the top of the plug.

Wilson Plug

Theorem: [Wilson, 1966] A closed oriented 3-manifold M admits a smooth non-vanishing vector field $\mathcal{X}$ with two periodic orbits.

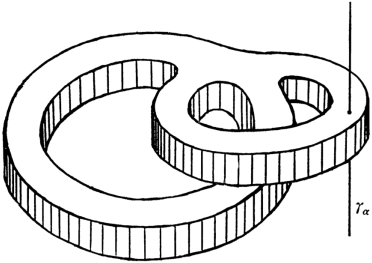
Proof: Trap orbits so they limit to a periodic orbit contained in the plug. The two periodic orbits are attractors:



Schweitzer Plug

Theorem: [Schweitzer, 1974] Every homotopy class of non-vanishing vector fields on a closed 3-manifold M contains a C^1-vector field without closed orbits.

Proof: Replace the circular orbits of the Wilson Plug with Denjoy minimal sets, embedded as pictured, so trapped orbits limit to Denjoy minimal set.



Handel's Theorem

A minimal set K for a flow $\mathcal{X}$ on a 3-manifold M is said to be "surface-like" if there is a tamely embedded surface $\Sigma \hookrightarrow M$ whose image contains K.

Theorem: [Handel, 1980] Let $\mathcal{X}$ be a flow on a 3-manifold M such that its minimal sets are surface-like, then $\mathcal{X}$ cannot be C^2.

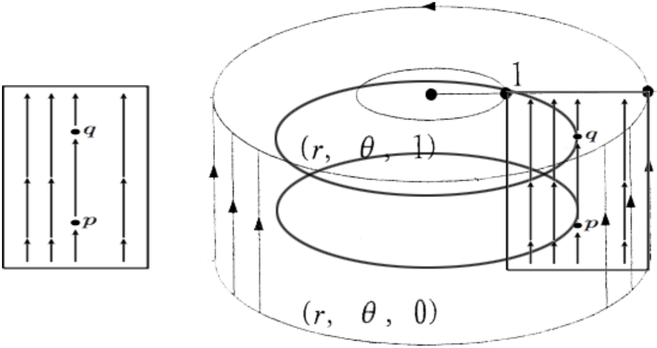
Handel's analysis implies any construction of counter-examples to the Seifert Conjecture requires "3-dimensional dynamics", in that its minimal sets cannot be planar.

Modified Wilson Plug

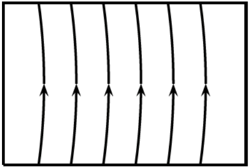
Define a radially symmetric vector field on the plug $\mathbb{W}$

$$\mathcal{W} = g(r,\theta,z)\partial/\partial z + f(r,\theta,z)\partial/\partial\theta$$

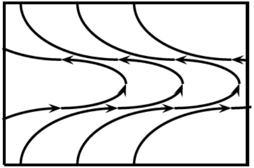
$(r,\theta,z) \in [1,3] \times \mathbb{S}^1 \times [-2,2] = W$ is radially symmetric, with $f(r,0) = 0$ and $g(r,z) = 0$ only near the boundary, and $g(r,z) = 1$ away from the boundary ∂W.



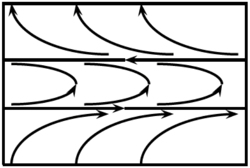
The flattened orbits of the modified Wilson flow $\mathcal{W}$ appear like:



$r \approx 1, 3$

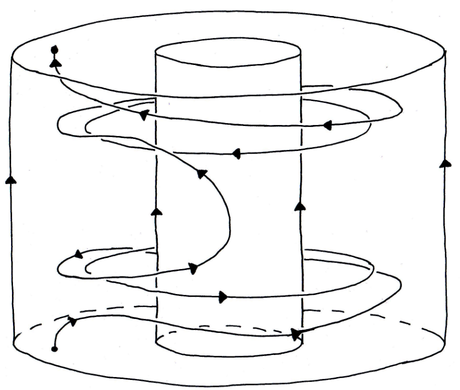


$r \approx 2$

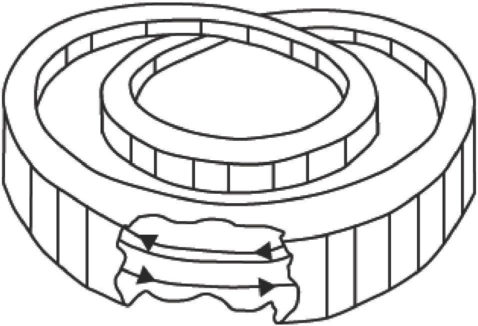


$r = 2$

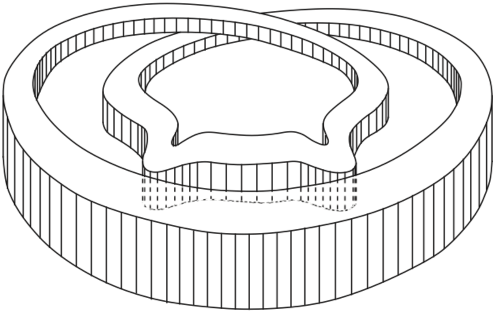
3d-orbits of $\mathcal{W}$ appear for $r > 2$ appear like:



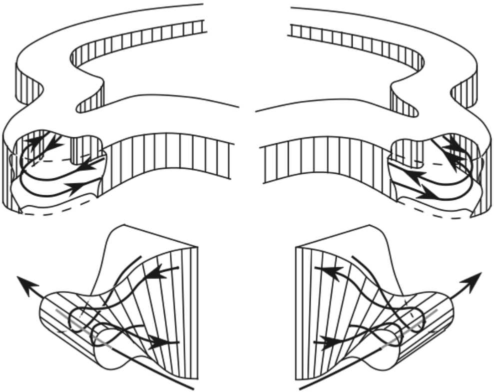
Embed modified Wilson as a double cover



Grow two horns

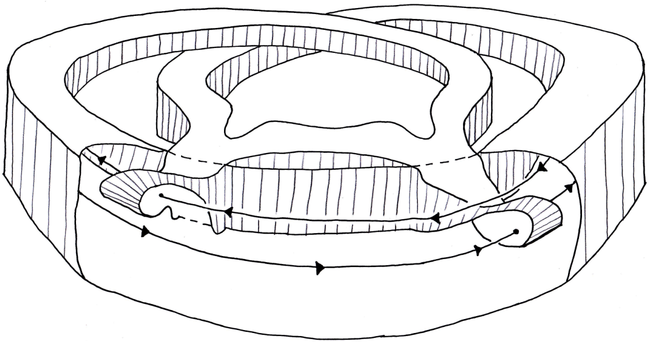


Twist the horns

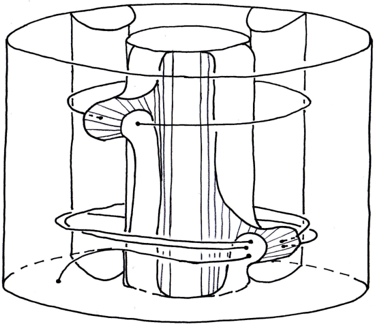


Insert the horns

The vector field $\mathcal{W}$ induces a field $\mathcal{K}$ on the surgered manifold. Then the Kuperberg Plug is pictured as:



Wilson dynamics + insertions = Kuperberg dynamics



This is an aperiodic plug, as only chance for periodic orbit is via the circular Wilson orbits, and they get broken up.

Kuperberg Plug

Shigenori Matsumoto's summary:

そこで，どうしても W 内のふたつの周期軌道 T_1 と T_2 を予め破壊しておく必要がある．しかしそのために新しい部品を開発するのでは話は振り出しに戻ってしまう．Kuperberg の発想は，W 内の**周期軌道自身で自分達を破壊させる**というものである．敵同士が妨害工作をしあうようにわなを仕掛けた後は，何もせずに黙ってみていればよいということである．

We therefore must demolish the two closed orbits in the Wilson Plug beforehand. But producing a new plug will take us back to the starting line. The idea of Kuperberg is to let closed orbits demolish themselves. We set up a trap within enemy lines and watch them settle their dispute while we take no active part.

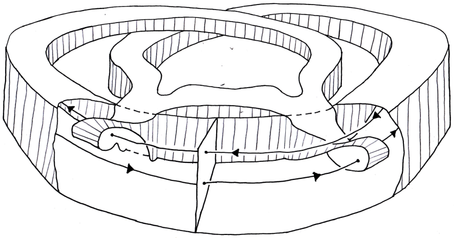
(transl. by Kiki Hudson Arai)

The transverse section

Consider a rectangular section

$$\mathbf{R}_0 = \{(r, \pi, z) \mid 1 \leq r \leq 3 \ \& \ -2 \leq z \leq 2\}$$

as pictured here, which is disjoint from insertions. The periodic orbits in $\mathbb{W}$ intersect this in two special points $\omega_1 = (2, \pi, -1)$ and $\omega_2 = (2, \pi, 1)$.



The minimal set

$\Phi_t : \mathbb{K} \to \mathbb{K}$ is Kuperberg flow.

- Every trapped orbit in $\mathbb{K}$ contains either ω_1 or ω_2 in its closure.

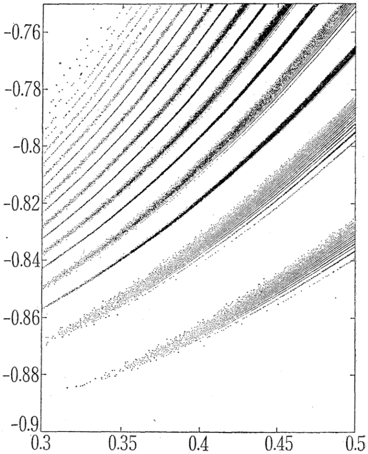
Define the orbit closures $\Sigma_i = \overline{\{\Phi_t(\omega_i) \mid t \in \mathbb{R}\}}$ for $i = 1, 2$.

Theorem: $\Sigma_1 = \Sigma_2$ and $\Sigma \equiv \Sigma_1$ is the unique minimal set for Φ_t.

What is the topological shape of Σ?

What is the Hausdorff dimension of the intersection $\Sigma \cap \mathbf{R}_0$?

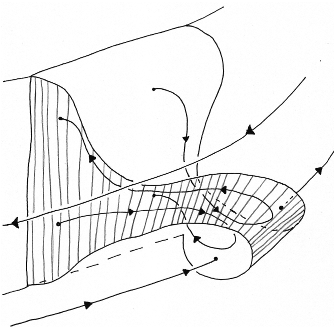
Suggestive computer illustration of $\Sigma \cap \mathbf{R}_0$, by B. Sévennec (1994).

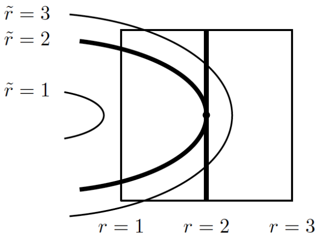


Quadratic hypotheses

A more detailed study of the Kuperberg dynamics requires some type of regularity hypotheses on the construction.

Definition: A Kuperberg flow $\mathcal{K}$ is said to be *generic* if the singularities for the vanishing of the vertical part $g(r, \theta, z)\frac{\partial}{\partial z}$ of the Wilson vector field $\mathcal{W}$ are of quadratic type, and each insertion map σ_i for $i = 1, 2$ yields a quadratic radius function.



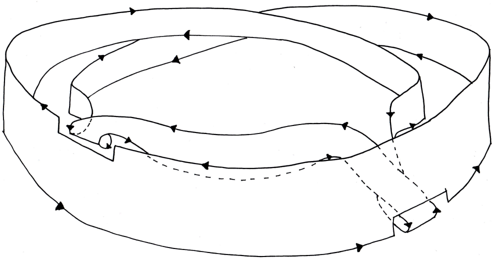


The twisted band and the propellers

Reeb cylinder $\mathcal{R} = \{(2, \theta, z) \mid 0 \le \theta \le 2\pi \ \& \ -1 \le z \le 1\} \subset \mathbb{W}$

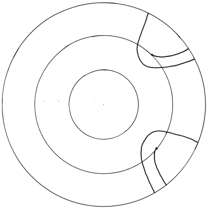
$\mathcal{R}'$ is Reeb cylinder minus insertions. Introduce Φ_t-invariant sets

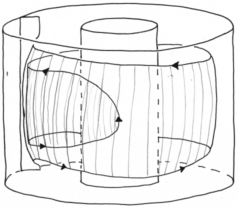
$$\mathfrak{M}_0 = \bigcup_{t \in \mathbb{R}} \Phi_t(\mathcal{R}') \quad , \quad \mathfrak{M} \equiv \overline{\mathfrak{M}_0} \subset \mathbb{K}$$

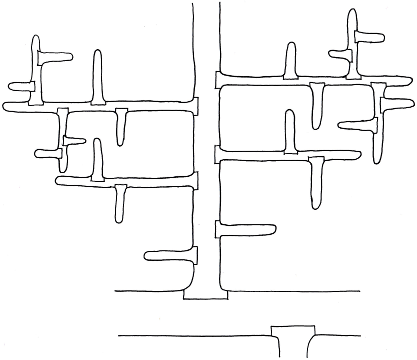


Levels

$$\mathfrak{M}_0 = \mathcal{R}' \cup \mathfrak{M}_0^1 \cup \mathfrak{M}_0^2 \cup \cdots$$







Theorem: [H & R] For the generic Kuperberg flow, the minimal set $\mathcal{Z} = \mathfrak{M}$, and thus is 2-dimensional.

$$\mathcal{R}' \subset \mathcal{Z} = \overline{\{\Phi_t(\omega_i) \mid t \in \mathbb{R}\}} \subset \overline{\bigcup_{t \in \mathbb{R}} \Phi_t(\mathcal{R}')} = \mathfrak{M}.$$

Proof: Show that the orbit $\Phi_t(\omega_1)$ forms a spiral about $\mathcal{R}'$ which is uniformly spaced tending to zero as $r \to 2$.

The two papers below discuss the existence of open disks in Σ.

- É. Ghys, "Construction de champs de vecteurs sans orbite périodique (d'après Krystyna Kuperberg)", Séminaire Bourbaki, Vol. 1993/94, Exp. No. 785, 1995.
- G. Kuperberg and K. Kuperberg, "Generalized counterexamples to the Seifert conjecture" , Annals of Math, 1996.

Wandering points

Theorem: For the generic Kuperberg flow, all points in the complement $\mathbb{K} - \mathfrak{M}$ are wandering. Moreover, any orbit which is entirely contained in either the region for $r > 2$, or the region $r < 2$ of $\mathbb{K}$, cannot be infinite.

Proof: Combine results of Ghys and Matsumoto for orbits with $r \leq 2$, and the authors for the case where $r > 2$.

Conclusion: $\mathfrak{M}$ is where the dynamical properties of the Kuperberg flow are determined.

Zippered laminations

Theorem:[H & R] For the generic Kuperberg flow:

• There is a lamination $\mathcal{L} \subset \mathfrak{M}$ with open leaves;

• Each leaf of $\mathcal{L}$ has a tree structure, with branching number at most 4;

• $\partial_z \mathfrak{M} = \mathfrak{M} - \mathcal{L}$ is the union of the boundaries of the leaves of $\mathcal{L}$;

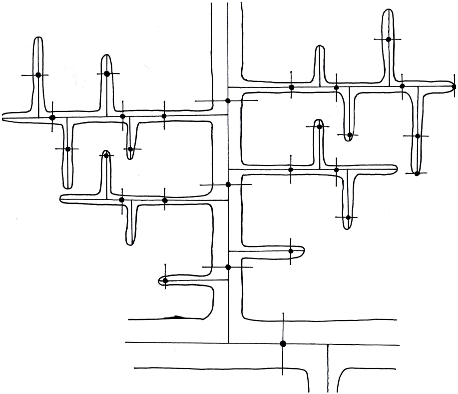
• $\partial_z \mathfrak{M}$ is dense in $\mathfrak{M}$.

So $\mathfrak{M}$ is like a lamination with boundary, except that its "boundary" $\partial_z \mathfrak{M}$ is dense in $\mathfrak{M}$. To describe the topological properties of $\mathfrak{M}$ we must recall how $\mathfrak{M}_0$ was defined.

Cantor transversal

$\mathcal{T} = \{(r, \pi, 0) \mid 1 \leq r \leq 3\} \subset \mathbf{R}_0$ is transverse to $\mathfrak{M}_0$.

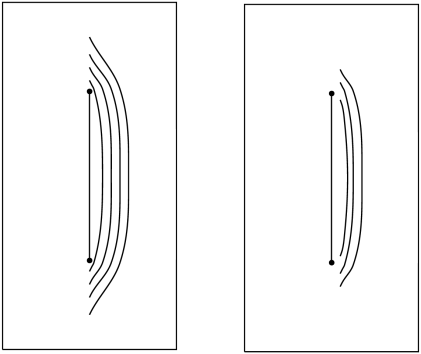
Theorem: $\mathfrak{C} = \mathfrak{M} \cap \mathcal{T}$ is a Cantor set, which is a complete transversal for the open leaves $\mathcal{L} \subset \mathfrak{M}$.



Transverse pseudogroup

The flow Φ_t induces a pseudogroup $\mathcal{G}_K$ on $\mathbf{R}_0$

Five generators $\{\psi, \phi_1^{\pm}, \phi_2^{\pm}\}$ – these act on curves in $\mathfrak{M}_0 \cap \mathbf{R}_0$ (illustrated below) to obtain $\mathcal{G}_{\mathfrak{M}}$ acting on $\mathfrak{C}$.



Lamination entropy

For $\epsilon > 0$, say that $\xi_1, \xi_2 \in \mathfrak{C}$ are (n, ϵ)-separated if there exists $\varphi \in \mathcal{G}_{\mathfrak{M}}^{(n)}$ with $\xi_1, \xi_2 \in Dom(\varphi)$, and $d_{\mathfrak{C}}(\varphi(\xi_1), \varphi(\xi_2)) \ \geq \ \epsilon$.

A finite set $\mathcal{S} \subset \mathfrak{C}$ is said to be (n, ϵ)-separated if every distinct pair $\xi_1, \xi_2 \in \mathcal{S}$ are (n, ϵ)-separated.

Let $s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)$ be the maximal cardinality of an (n, ϵ)-separated subset of $\mathfrak{M}$.

The *lamination entropy* of $\mathcal{G}_{\mathfrak{M}}$ is defined by:

$$h(\mathcal{G}_{\mathfrak{M}}) \;=\; \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{1}{n} \ln(s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)) \right\}.$$

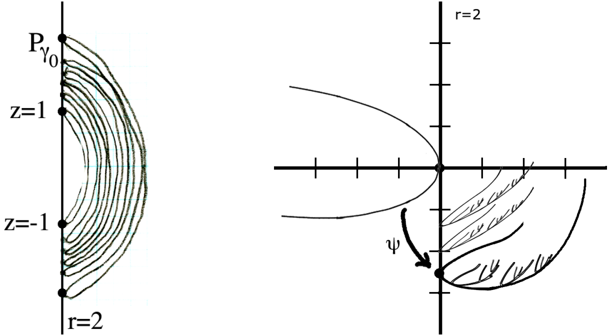
The limit $h(\mathcal{G}_{\mathfrak{M}})$ depends on the generating set, but the fact of being non-zero does not.

For $0 < \alpha < 1$, define the *slow entropy* for $\mathcal{G}_{\mathfrak{M}}$

$$h_{\alpha}(\mathcal{G}_{\mathfrak{M}}) \;=\; \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{1}{n^{\alpha}} \ln(s(\mathcal{G}_{\mathfrak{M}}, n, \epsilon)) \right\}.$$

Theorem: (H and R) For $\alpha = 1/2$, the pseudogroup $\mathcal{G}_{\mathfrak{M}}$ has positive slow entropy, $h_{1/2}(\mathcal{G}_{\mathfrak{M}}) > 0$.

Proof: The parabolic arcs are one-half of a stretched ellipse, and the action of $\mathcal{G}_K$ maps ellipses into ellipses, so can get a good game of ping-pong up and running. The only catch is that the game runs slow. The n^{th}–volley takes approximately n^2 steps.



Sullivan dictionary

Remark: The entropy $h(\Phi_t)$ of the flow Φ_t is calculated by following the path around the perimeter of the leaf $\mathfrak{M}_0$, while for the entropy $h(\mathcal{G}_{\mathfrak{M}})$ we are allowed to follow straight paths, along the Cayley graph of $\mathcal{G}_{\mathfrak{M}}$ in $\mathfrak{M}_0$.

The same points get separated, but just at different rates.

Analogous to behavior of horocycle flow (Kuperberg flow Φ_t) verses geodesic flow (Wilson flow Ψ_t), even including "nested ellipses at infinity" in $\mathbf{R}_0$.

Shape of the minimal set

For $\epsilon > 0$, let $N_\epsilon(\mathfrak{M}) = \{x \in \mathbb{K} \mid d(x, \mathfrak{M}) < \epsilon\}$.

Definition: The shape of $\mathfrak{M}$ is $\mathcal{S}(\mathfrak{M}) = \varprojlim \; \{N_\epsilon \mid \epsilon > 0\}$.

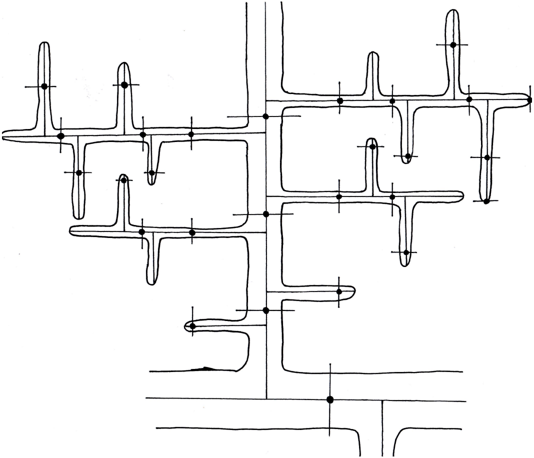
Definition: A continua $K \subset M$ is said to have *stable shape* if there exists a cofinal sequence $\epsilon_\ell > 0$ for $\ell \geq 1$, such that $\epsilon_\ell \to 0$, and the inclusion $N_{\epsilon_\ell} \subset N_{\ell_1}$ is a homotopy equivalence for all $\ell \geq \ell_1$.

Example: The Denjoy minimal set has stable shape $\cong \mathbb{S}^1 \vee \mathbb{S}^1$.

We then realize the original motivation for this work:

Theorem: [H & R] The shape $\mathcal{S}(\mathfrak{M})$ is not stable.

Proof: The shape homotopy group $\pi_1^{sh}(\mathfrak{M}, \omega_1)$ is generated by paths in $\mathfrak{M}_0$ going out to the ends of the embedded tree, so is not stable!



Remarks and Problems

Theorem: [H & R] In every C^1-neighborhood of $\mathcal{K}$, there exists a smooth flow Φ'_t on $\mathbb{K}$ with positive entropy, and the associated invariant lamination $\mathfrak{M}'$ is the suspension of a horseshoe. There also exists a smooth flow Φ''_t on $\mathbb{K}$ with no trapped orbits.

Problem: Give a description of the dynamical properties of all flows C^1-close to a generic Kuperberg flow.

Theorem: [Kuperbergs, 1996] There exists a PL flow on $\mathbb{K}$ for which the minimal set Σ is 1-dimensional.

Problem: Characterize the smooth (non-generic!) flows on $\mathfrak{M}$ for which Σ is 1-dimensional.

Problem: Understand the topological orbit-equivalence class of the action $\mathcal{G}_{\mathfrak{M}}$ on $\mathfrak{C}$.

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