# Dynamics of Group Actions and Minimal Sets

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# Abstract:


Ongoing joint works with Alex Clark & Olga Lukina, Leicester University, UK.

The study of foliation dynamics aims to understand the asymptotic properties of its leaves, and identify geometric and topological "structures" which are associated to them; e.g., the minimal sets of the foliation.

The dynamics of a foliation partitions the ambient manifold into three disjoint saturated Borel sets: the Elliptic, Parabolic and Hyperbolic regions. A fundamental open problem is to describe the properties of minimal sets contained in each of these regions.

Alex Clark and the author showed that there exists smooth actions of $\mathbb{Z}^n$ with a continuum of distinct minimal sets, all contained in the union of elliptic and parabolic sets, and no two of which are homeomorphic. These minimal sets are "weak solenoids", and give rise to a continuum of secondary invariants.

The study of these examples leads to the more general study of properties and classification of matchbox manifolds, a particular class of continua that arise in dynamical systems.

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- Is the system structurally stable under $C^r$-perturbations, $r \geq 1$?
- Find cohomology invariants of the system which characterize it.

Also consider non-singular vector field $\vec{X}$ on a closed manifold $M$ which defines a 1-dimensional foliation $\mathcal{F}$ on $M$.

Smale also suggested to study these points for large group actions.

Students of Godbillon, Smale, Tamura studied foliation dynamics.

# Foliation dynamics

A foliation $\mathcal{F}$ of dimension $n$ on a smooth manifold $M^m$ decomposes the space into "uniform layers" – the leaves.

$M$ is a $C^r$ *foliated manifold* if the transition functions for the foliation charts $\varphi_i \colon U_i \to [-1,1]^n \times T_i$ (where $T_i \subset \mathbb{R}^q$ is open) are $C^\infty$ leafwise, and vary $C^r$ with the transverse parameter in the leafwise $C^\infty$-topology.

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For a continuous dynamical system on a compact manifold $M$ defined by a flow $\varphi \colon M \times \mathbb{R} \to M$, the orbit $L_x = \{\varphi_t(x) = \varphi(x,t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$.

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Foliation dynamics: replace the concept of time-ordered trajectories with multi-dimensional futures for points; then study the aggregate and statistical behavior of the collection of its leaves.

# Group actions

$\Gamma = \langle \gamma_1, \ldots, \gamma_d \rangle$ is a finitely generated group.

$\varphi \colon \Gamma \times N \to N$ is $C^r$-action on closed manifold of dimension $q$, $r \geq 1$.

If $\Gamma \cong \pi_1(B, b_0)$ and $\widetilde{B} \to B$ is the universal covering, then

$$M = (\widetilde{B} \times N)/\Gamma \to B$$

is a foliated bundle, where the transverse holonomy of $\mathcal{F}_\varphi$ determines the action $\varphi$ up to conjugacy.

$$\text{Dynamics of action } \varphi \quad \Longleftrightarrow \quad \text{Dynamics of leaves of } \mathcal{F}_\varphi$$

Each point of view has advantages, limitations.

# Pseudogroups

A section $\mathcal{T} \subset M$ for $\mathcal{F}$ is an embedded submanifold of dimension $q$ which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ on $\mathcal{T}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$.

**Definition:** A pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if there is

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}$ such that $\langle \Gamma \rangle = \mathcal{G}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

## Groupoid word length

**Definition:** The groupoid of $\mathcal{G}$ is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \ \& \ x \in D(g)\} \ , \ \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

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For $g \in \Gamma_{\mathcal{G}}$, the *word length* $\|[g]\|_x$ of the germ $[g]_x$ of g at $x$ is the least $k$ such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_k}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another along an orbit or leaf, while preserving the germinal dynamics.

# Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

**Definition:** The normal cocycle $D\varphi \colon \Gamma_{\mathcal{G}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D\varphi[g]_x = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D[h]_y \cdot D[g]_x$$

# Asymptotic exponent – foliations

**Definition:** The transverse expansion rate function at $x$ is

$$\lambda(\mathcal{G}, k, x) = \max_{\|[g]\|_x \leq k} \frac{\ln\left(\max\{\|D_x g\|, \|(D_y g^{-1}\|\}\right)}{k} \geq 0$$

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This is essentially the "maximum Lyapunov exponent" for $\mathcal{G}$ at $x$.

$\lambda(\mathcal{G}, x)$ is a Borel function of $x \in \mathcal{T}$, as each norm function $\|D_{w'} h_{\sigma_{w,z}}\|$ is continuous for $w' \in D(h_{\sigma_{w,z}})$ and the maximum of Borel functions is Borel.

**Lemma:** $\lambda_{\mathcal{F}}(z)$ is constant along leaves of $\mathcal{F}$.

# Asymptotic exponent – group actions

Let $\varphi \colon \Gamma \times N \to N$ be a $C^1$-action.

**Definition:** The transverse expansion rate function at $x$ is

$$\lambda(\varphi, k, x) = \max_{\|\gamma\| \leq k} \frac{\ln \left( \max\{\|D_x\varphi(\gamma)\|, \|(D_y\varphi(\gamma)^{-1}\|\}\right)}{k} \geq 0$$

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This is essentially the "maximum Lyapunov exponent" for $\mathcal{G}$ at $x$ – for all $x \in N$, $\epsilon > 0$, there exists a sequence $\{\gamma_\ell \in \Gamma \mid \|\gamma\| \to \infty\}$,

$$\max\left\{\|D_x\varphi(\gamma_\ell)\|, \|(D_y\varphi(\gamma_\ell)^{-1}\|\right\} \geq \exp\left\{\ell \cdot (\lambda(\varphi, k, x) - \epsilon)\right\}$$

# Expansion classification

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3. Partially Hyperbolic points: $\mathcal{H} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}, x) > 0\}$
   i.e., "points of exponential-growth expansion" –
   non-uniformly, partially hyperbolic foliations

# Minimal and transitive sets

$M$ compact foliated, $\mathcal{F}$ foliation of codimension-$n$. $\mathfrak{M} \subset M$ is

- minimal if it is closed, $\mathcal{F}$-saturated, and every leaf in $\mathfrak{M}$ is dense.
- transitive if it is closed, $\mathcal{F}$-saturated, and there exists a dense leaf in $\mathfrak{M}$.

**Remark:** A minimal set $\mathfrak{M}$ is an example of a *continuum*; that is, a compact and connected metrizable space. In fact, it is an *indecomposable continuum*, which is a continuum that is not the union of two proper subcontinua.

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For a group action $\varphi$, a minimal set $\mathfrak{M}$ for $\mathcal{F}_\varphi$ is the compactification of $\Gamma$ associated to the sub $C^*$-algebra $\varphi_x^* \colon C^0(N) \to C_b(\Gamma)$.

# Shape dynamics of minimal sets

The *shape* of a minimal set $\mathfrak{M}$ is defined by a co-final descending chain $\{U_\ell \mid \ell \geq 1\}$ of open neighborhoods

$$U_1 \supset U_2 \supset \cdots \supset U_\ell \supset \cdots \supset \mathfrak{M} \quad ; \quad \bigcap_{\ell=1}^{\infty} U_\ell = \mathfrak{M}$$

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The *shape dynamics* of $\mathfrak{M}$ is the germ of the dynamical system $\mathcal{F}$ defined by a shape approximation to $\mathfrak{M}$. This is equivalent to specifying the *foliated microbundle* of $\mathcal{F}$ defined by $\mathfrak{M} \subset M$.

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**Definition:** The shape of $\mathfrak{M}$ is *stable* if there exists $\ell_0$ such that for $\ell \geq \ell_0$ the inclusion $\mathfrak{M} \subset U_{\ell+1} \subset U_\ell$ is a homotopy equivalence.

# Remarks and questions – codimension-one

The first question is very old:

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**Question:** What are the minimal sets in a codimension-1, $C^r$-foliation?

**Example:** Denjoy minimal sets for $C^1$-foliations are stable.

**Example:** Markov minimal sets for $C^2$-foliations are stable, but this need not be true for $C^1$.

# Remarks and questions – codimension-$q > 1$

**Question:** What indecomposable continua can arise as minimal sets in codimension-$q$? Are there restrictions on their shape types for $C^r$-dynamics, depending on $r > 0$?

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The Sierpinsky torus $\mathbb{T}^q$ and its generalizations can be realized as hyperbolic minimal sets, $q > 1$.

**Question:** Are there conditions on the shape dynamics of $\mathfrak{M}$ which force $\mathfrak{M}$ to have stable shape? (e.g., hyperbolicity)

# Matchbox manifolds

**Definition:** An $n$-dimensional *matchbox manifold* is a continuum $\mathfrak{M}$ which is a foliated space with codimension zero and leaf dimension $n$. Essentially, same concept as laminations.

$\mathfrak{M}$ is a foliated space if it admits a covering $\mathcal{U} = \{\varphi_i \mid 1 \leq i \leq \nu\}$ with foliated coordinate charts $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$. The compact metric spaces $\mathfrak{T}_i$ are totally disconnected $\iff \mathfrak{M}$ is a matchbox manifold.

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Leaves of $\mathcal{F}$ $\Longleftrightarrow$ path components of $\mathfrak{M}$ $\Longleftrightarrow$ proper subcontinua

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, with the transverse dynamics of the foliation $\mathcal{F}$ on the Cantor-like fibers $\mathfrak{T}_i$ representing fundamental groupoid data.

# Embedding matchbox manifolds

**Problem:** Let $\mathfrak{M}$ be a minimal matchbox manifold of dimension $n$. When does there exists a $C^r$-foliation $\mathcal{F}_M$ of a compact manifold $M$ and a foliated topological embedding $\iota\colon \mathfrak{M} \to M$ realizing $\mathfrak{M}$ as a minimal set?

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The space of tilings associated to a given quasi-periodic tiling of $\mathbb{R}^n$ is a matchbox manifold. For a few classes of quasi-periodic tilings of $\mathbb{R}^n$, the codimension one canonical cut and project tiling spaces, it is known that the associated matchbox manifold is a minimal set for a $C^1$-foliation of a torus $\mathbb{T}^{n+1}$, where the foliation is a generalized Denjoy example.

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The "Williams solenoids", introduced by Bob Williams in 1967, 1974 as the attractors of certain Axiom A systems, are matchbox manifolds. It is unknown which of the Williams solenoids can be embedded as minimal sets for foliations of closed manifolds.

# Topological dynamics

**Definition:** $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \quad \Longrightarrow \quad d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

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**Theorem:** [Clark-Hurder 2010] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal.

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**Theorem:** [Clark-Hurder 2010] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal.

**Definition:** $\mathfrak{M}$ is an *expansive matchbox manifold* if it admits some covering by foliation charts as above, such that there exists $\epsilon > 0$, so that for all $x \neq x' \in \mathcal{T}$ with $d_{\mathcal{T}}(x, x') < \epsilon$, there exists $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ such that

$$d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(x')) \geq \epsilon$$

# Weak solenoids

Let $B_\ell$ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

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The $p_\ell$ are called the *bonding maps* for the weak solenoid

$$\mathcal{S} = \lim_{\leftarrow} \; \{p_\ell \colon B_\ell \to B_{\ell-1}\} \; \subset \; \prod_{\ell=0}^{\infty} B_\ell$$

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Choose basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$.

Set $G_\ell = \pi_1(B_\ell, x_\ell)$.

# McCord solenoids

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

**Definition:** $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of $G_{\ell_0}$.

**Theorem** [McCord 1965] Let $B_0$ be an oriented smooth closed manifold. Then a McCord solenoid $\mathcal{S}$ is an orientable, homogeneous, equicontinuous smooth matchbox manifold.

# Classifying weak solenoids

A weak solenoid is determined by the base manifold $B_0$ and the tower equivalence of the descending chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

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**Theorem:**[Pontragin 1934; Baer 1937] For $G_0 \cong \mathbb{Z}$, the homeomorphism types of McCord solenoids is uncountable.

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**Theorem:**[Pontragin 1934; Baer 1937] For $G_0 \cong \mathbb{Z}$, the homeomorphism types of McCord solenoids is uncountable.

**Theorem:**[Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^k$ with $k \geq 2$, the homeomorphism types of McCord solenoids is not classifiable, *in the sense of Descriptive Set Theory*.

The number of such is not just huge, but indescribably large.

# Homogeneous matchbox manifolds

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**Corollary:** Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to the suspension of an minimal action of a countable group on a Cantor group $\mathbb{K}$.

# Embedding results

**Problem:** Let $r \geq 0$. What types of towers of finitely-generated groups

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

arise from equicontinuous minimal sets of $C^r$-foliations?

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It seems that extremely little is known about such questions.

We present some results for the case $\Gamma = \mathbb{Z}^k$.

Note that solenoids do not have stable shape.

# Topological embeddings

Our strongest results are for $C^0$-embedding problem – every presentation of a solenoid with base $\mathbb{T}^k$ admits an embedding into a $C^0$-foliation.

**Theorem:** [Clark & Hurder] Let $\mathcal{P}$ be a presentation of the solenoid $\mathcal{S}$ over the base space $\mathbb{T}^k$, and let $q \geq 2k$. Then there exists a $C^0$-foliation $\widehat{\mathcal{F}}$ of $\mathbb{T}^k \times \mathbb{D}^q$ such that:

1. $\widehat{\mathcal{F}}$ is a distal foliation, with smooth transverse invariant volume form;
2. $L_0 = \mathbb{T}^k \times \{\vec{0}\}$ is a leaf of $\widehat{\mathcal{F}}$, and $\widehat{\mathcal{F}} = \mathcal{F}_0$ near the boundary of $M$;
3. there is an embedding of $\mathcal{P}$ into the foliation $\widehat{\mathcal{F}}$;
4. the solenoid $\mathcal{S}$ embeds as a minimal set $\widehat{\mathcal{F}}$.

## Smooth embeddings

The embedding problem for solenoids into $C^1$-foliations is the next most general case.

**Theorem:** [Clark & Hurder] Let $\mathcal{P}$ be a presentation of the solenoid $\mathcal{S}$ over the base space $\mathbb{T}^k$, and let $q \geq 2k$. Suppose that $\mathcal{P}$ admits a sub-presentation $\mathcal{P}'$ which satisfies condition (**). Then there exists a $C^1$-foliation $\widehat{\mathcal{F}}$ of $\mathbb{T}^k \times \mathbb{D}^q$ such that:

1. $\widehat{\mathcal{F}}$ is a distal foliation, with smooth transverse invariant volume form;
2. $L_0 = \mathbb{T}^k \times \{\vec{0}\}$ is a leaf of $\widehat{\mathcal{F}}$, and $\widehat{\mathcal{F}} = \mathcal{F}_0$ near the boundary of $M$;
3. there is an embedding of $\mathcal{P}'$ into the foliation $\widehat{\mathcal{F}}$;
4. the solenoid $\mathcal{S}$ embeds as a minimal set $\widehat{\mathcal{F}}$.

## anti-Reeb-Thurston Stability

**Theorem:** [Clark & Hurder] Let $\mathcal{F}_0$ be a $C^\infty$-foliation of codimension $q \geq 2$ on a manifold $M$. Let $L_0$ be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some saturated open neighborhood $U$ of $L_0$. Then there exists a foliation $\mathcal{F}_M$ on $M$ which is $C^\infty$-close to $\mathcal{F}_0$, and $\mathcal{F}_M$ has an uncountable set of solenoidal minimal sets $\{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$, all contained in $U$, and *pairwise non-homeomorphic*.

If $\mathcal{F}_0$ is a distal foliation with a smooth transverse invariant volume form, then the same holds for $\mathcal{F}_M$.

## Get a grip - some open problems

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**Problem:** Suppose that $\mathfrak{M}$ is homeomorphic to a minimal set in a $C^2$-foliation, and the transversals are $k$-connected, for $0 \leq k < q$. Are there examples besides Sierpinski manifolds, suspensions of minimal actions on Cantor groups, and various products of these? Is there a possible structure theory ?