Applications of the Contraction Mapping Theorem: Circle actions

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Definition. Let (X, d) be a metric space. Then a map $T \colon X \to X$ is called a *contraction mapping* on X if there exists $0 \leq C < 1$ such that $d(T(x), T(y)) \leq C \cdot d(x, y)$ for all $x, y \in X$.

Banach Fixed-Point Theorem. [Banach, 1922] Let (X, d) be a non-empty complete metric space with a contraction mapping $T \colon X \to X$. Then T admits a unique fixed-point $x^* \in X$.

In these lectures, we give three applications of this theorem to the study of group actions on compact spaces.

- Γ will always denote a finitely generated group.

Question: What are the non-trivial actions of Γ on the circle $\mathbb{S}^1$?

Theorem. [Margulis] Let a group Γ act by homeomorphisms on $\mathbb{S}^1$. Then, either there is a Γ-invariant probability measure on $\mathbb{S}^1$, or Γ contains a free nonabelian group.

$\star$ Gregory Margulis, *Free subgroups of the homeomorphism group of the circle*, **C. R. Acad. Sci. Paris Sér. I Math.**, 331, 669–674,1980.

The proof uses a "local contraction principle" on the space of probability measures on $\mathbb{S}^1$ to construct an invariant Cantor set, and a ping-pong game for the action on this Cantor set.

Theorem. [Witte-Morris] Let Γ be an arithmetic subgroup of a $\mathbb{Q}$-simple algebraic $\mathbb{Q}$-group G, with $\mathbb{Q}-\mathrm{rank}(G) \geq 2$. Then every continuous action of Γ on the circle $\mathbb{S}^1$ or the real line $\mathbb{R}$ factors through the action of a finite quotient of Γ.

Example. Let $\Gamma \subset SL(n,\mathbb{R})$ be a lattice for $n \geq 3$, then an action of Γ on $\mathbb{S}^1$ factors through a finite action.

$\star$ Dave Witte, *Arithmetic groups of higher* **Q***-rank cannot act on 1-manifolds*, **Proc. Amer. Math. Soc.**, 122, 669–674, 1994.

The proof uses the *antithesis* of a contraction mapping principle, as it shows that a $\mathbb{Z}$-extension of Γ is left orderable.

Definition. A group Γ is called (left) orderable if there exists a total ordering $\leq$ on Γ which is compatible with the group law, in the sense that $g \leq h$ and $k \in \Gamma$ implies $k \cdot g \leq k \cdot h$.

Theorem. Γ acts faithfully on $\mathbb{R}$ if and only if Γ is left orderable.

Example. Every torsion free nilpotent group Γ acts faithfully on $\mathbb{R}$.

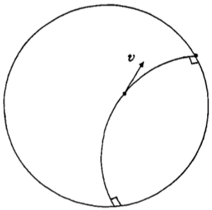
$\star$ Andrés Navas, **Groups of circle diffeomorphisms**, Chicago Lectures in Mathematics, University of Chicago Press, 2011.

Let $\Gamma \subset PSL(2, \mathbb{R})$ be a torsion-free cocompact subgroup; so a discrete subgroup such that the quotient $M = PSL(2, \mathbb{R})/\Gamma$ is a compact 3-manifold.

Let $\mathbb{D}^2 \subset \mathbb{C}^2$ be the unit disk with the Poincaré metric,

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2)^2)}$$

The geodesics for this model are the circular acrs



For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the action on $z = x + iy \in \mathbb{C}$ by $A \cdot z = \dfrac{az+b}{cz+d}$ maps the boundary $\mathbb{S}^1 = \{\|z\| = 1\}$ to itself, so defines an action $\Phi_0 \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$.

Alternately, each point $z \in \mathbb{S}^1$ is identified with the point at $+\infty$ for a family of geodesics in $\mathbb{D}^2$, and the action Φ_0 is described by its action on geodesics in $\mathbb{D}^2$.

Then the quotient $\Sigma = \mathbb{D}^2/\Gamma$ is a compact Riemann surface with a metric g_0 of constant negative curvature.

Consider a metric g on Σ with negative curvature - for example a metric which is C^3-close to g_0,

The lift $\widetilde{g}$ of g to the universal cover $\mathbb{D}^2 \to \Sigma$ is Γ-invariant, so the deck transformations preserve the geodesics of $\widetilde{g}$ in $\mathbb{D}^2$. The action of Γ then acts on the endpoints of the geodesics at $+\infty$, so defines an action $\Phi_g \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$. Let $\mathfrak{M}^-$ denote the space of metrics of negative curvature on Σ. Thus we obtain

$$\Phi \colon \mathfrak{M}^- \to \mathbf{Hom}\{\Gamma \to \mathbf{Homeo}(\mathbb{S}^1)\}$$

where $\Phi(g) = \Phi_g$.

The geodesic flow $\phi_t \colon M \to M$ for the Riemann surface Σ of constant negative curvature has a simple description when lifted to $SL(2, \mathbb{R})$, given by

$$\widetilde{\phi}_t \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e^{t/2}a & e^{t/2}b \\ e^{-t/2}c & e^{-t/2}d \end{bmatrix}$$

The action $\widetilde{\phi}_t$

- expands the subgroup $U^+ = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$
- contracts the subgroup $U^- = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$

The infinitesimal model for this action is given by the commutator actions on Lie the algebra $\mathfrak{sl}(2,\mathbb{R}) = \begin{bmatrix} r & s \\ t & -r \end{bmatrix}$, $r, s, t \in \mathbb{R}$

Let $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

then $[Z, X] = 2X$, $[Z, Y] = -2Y$ and $[X, Y] = Z$.

Then we have two Lie subalgebras $L^+ = \langle Z, X \rangle$ and $L^- = \langle Z, Y \rangle$ which generate subgroups of $SL(2,\mathbb{R})$. The left translates of these subgroups define codimension one foliations $\widetilde{\mathcal{F}}^+$ and $\widetilde{\mathcal{F}}^-$ which are invariant under the left action of Γ, so descend to foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ on M. Moreover, they are transverse to the $\mathbb{S}^1$ subgroup of $SL(2,\mathbb{R})$ generated by the Lie vector $\vec{\theta} = X - Y$.

Conclusion. There is a direct sum decomposition $TM = \langle Z \rangle \oplus \langle X \rangle \oplus \langle Y \rangle$ where the flow of ϕ_t is:

- tangent to the distribution $\langle Z \rangle$
- expands the distribution $\langle X \rangle$
- contracts the distribution $\langle Y \rangle$

Definition. Let M be a compact manifold of dimension n. A smooth flow ϕ_t of M is Anosov if there exists a direct sum decomposition $TM = E_0 \oplus E^+ \oplus E^-$ where the flow of ϕ_t is

- tangent to the distribution E_0
- uniformly expands the distribution E^+
- uniformly contracts the distribution E^-

Theorem. [Anosov] The *weak unstable* and *weak stable* distributions $E_0 \oplus E^+$ and $E_0 \oplus E^-$ are both integrable, and integrate to codimension one foliations denoted $\mathcal{F}^+$ and $\mathcal{F}^-$ on M.

Theorem. [Anosov] Anosov flows are structurally stable. If ϕ_t' is C^1-close to ϕ_t, then ϕ_t' is also an Anosov flow, and there is a homeomorphism of M conjugating orbits of ϕ_t to orbits of ϕ_t'.

⋆ D.V. Anosov, *Tangent fields of transversal foliations in U-systems*, **Math. Notes Acad. Sci., USSR**, 2:818–823, 1967.

⋆ D.V. Anosov, *Geodesic Flows on Closed Riemannian Manifolds with Negative Curvature*, 90, **Proc. Steklov Inst. Math.**, 1967.

The proof of structural stability by Anosov uses the *shadowing principle* for Anosov flows.

John Mather gave an alternate proof using the Banach Fixed-Point Theorem for the space of sections of a vector bundle.

⋆ Mather, *Anosov diffeomorphisms*, appendix to S. Smale, *Differentiable Dynamical Systems*, Bulleting A.M.S., 73, 1967, pages 792-795.

In Section 3 of work with Katok, we refined the argument of Mather to show the contraction mapping was locally defined on a space of sections with a norm defined by the Zygmund condition:

Definition. The Zygmund norm of $f\colon [a,b] \to \mathbb{R}$ is given by

$$\Lambda_*(f) = \sup_{a<x<b} \limsup_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{|h|}$$

Remark. $\Lambda_*(f) < \infty$ implies that f is α-Hölder for all $\alpha < 1$.

⋆ S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, **Publ. Math. Inst. Hautes Etudes Sci.**, 72:5–61, 1990.

Theorem. [Hurder-Katok] If ϕ_t is a volume preserving Anosov flow of a 3-manifold, then the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ are C^1 and their derivatives have bounded Zygmund norm.

Corollary. The induced action at infinity for a geodesic flow ϕ_t is $C^{1+\alpha}$ for all $\alpha < 1$. That is, the flow induces a homomorphism

$$\Phi \colon \Gamma \to \mathbf{Diff}^{1+\alpha}(\mathbb{S}^1) \subset \mathbf{Homeo}(\mathbb{S}^1)$$

$\Rightarrow$ variations of metrics of negative curvature on Σ generate variations of $C^{1+\alpha}$-actions of $\Gamma = \pi_1(\Sigma, x_0)$ on $\mathbb{S}^1$.

Let $\mathcal{F}$ be a codimension one C^2-foliation of a compact manifold M. Assume that the normal bundle to $\mathcal{F}$ is orientable, then there exists a 1-form ω such that $\ker(\omega) = T\mathcal{F}$.

By the Frobenius Theorem, the distribution $T\mathcal{F}$ is integrable implies that there is a 1-form η such that $d\omega = \eta \wedge \omega$. Then a calculation shows that $\eta \wedge d\eta$ is closed, and that its cohomology class $[\eta \wedge d\eta] \in H^3(M; \mathbb{R})$ is independent of the choice of ω.

Theorem. [Godbillon-Vey] The class $GV(\mathcal{F}) \in H^3(M; \mathbb{R})$ is a diffeomorphism invariant of the foliation $\mathcal{F}$, and depends only on the concordance class of $\mathcal{F}$.

Roussarie calculated the Godbillon-Vey invariant for the weak unstable foliation of $M = SL(2, \mathbb{R})/\Gamma$.

Let X^*, Y^*, Z^* be the dual 1-forms for the framing $\{X, Y, Z\}$ of TM. Then $T\mathcal{F}^+ = \ker(Y^*)$ so take $\omega = Y^*$. Then $d\omega = -2Z^* \wedge Y^*$ and set $\eta = -2Z^*$, so we get

$$\eta \wedge d\eta = -2Z^* \wedge -2dZ^* = 4Z^* \wedge X^* \wedge Y^*$$

is a volume form on M. Hence $0 \neq GV(\mathcal{F}^+) \in H^3(M; \mathbb{R})$.

$\star$ C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension 1*, **C.R. Acad. Sci. Paris**, 273:92–95, 1971.

Suppose that $\mathcal{F}$ is a codimension one foliation on a circle bundle $\mathbb{S}^1 \to M \to B$ which is transverse to the circle fibers. Then there is a map, "integration over the fibers"

$$\int_{\mathbb{S}^1} : H^3(M; \mathbb{R}) \to H^2(B; \mathbb{R})$$

which yields a class $gv(\mathcal{F}) = \int_{\mathbb{S}^1} GV(\mathcal{F}) \in H^2(B; \mathbb{R})$.

Suppose the base $B = \Sigma$ as for the case $M = SL(2, \mathbb{R})/\Gamma$ then $gv(\mathcal{F}) \in H^2(\Sigma; \mathbb{R}) \cong H^2(\Gamma; \mathbb{R})$.

There is a formula for this class, given by Thurston:

Let $\gamma_1, \gamma_2 \in \Gamma$ and let $\alpha(\gamma_i) \in \textbf{Diff}(\mathbb{S}^1)$. Set

$$f = \log\{\alpha(\gamma_2)'\} \quad , \quad g = \log\{\alpha(\gamma_1\gamma_2)'\}$$

$$gv(\gamma_1, \gamma_2) = \int_{\mathbb{S}^1} f \; dg = \textbf{Area}\{(f, g) \colon \mathbb{S}^1 \to \mathbb{R}^2\}$$

This last equality follows from Stokes' Theorem for planar curves.

$\star$ R. Bott, *On some formulas for the characteristic classes of group actions*, in **Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976)**, Lect. Notes in Math. Vol. 652, 1978:25–61.

Observations.

• The area integral $\int_{\mathbb{S}^1} f\ dg$ is well defined when the image of $\mathbb{S}^1$ has Lebesgue measure 0.

• If the maps f, g are α-Hölder continuous for $\alpha > 1/2$ then the image circle in $\mathbb{R}^2$ has Lebesgue measure 0.

Theorem. [Hurder & Katok, 1991] Let g be a metric of negative curvature on a compact surface Σ. Then there is a well-defined Godbillon-Vey class $gv(\mathcal{F}_g^+) \in H^2(\Gamma; \mathbb{R})$.

$\star$ T. Tsuboi, *On the Hurder-Katok extension of the Godbillon-Vey invariant*, **J. Fac. Sci. Univ. Tokyo**, 37:255–262, 1990.

Theorem. [Formula of Mitsumatsu] Let (Σ, g) be a closed Riemann surface with strictly negative curvature and Euler characteristic $\chi(\Sigma)$. Then

$$\langle GV(\mathcal{F}_g^+), [M]\rangle = 4\pi^2 \cdot \chi(\Sigma) - 3 \cdot \int_M (\frac{\partial}{\partial\theta} H)^2 dvol$$

where $\frac{\partial}{\partial\theta}$ is the unit tangent vector to the circle fibers of $M \to \Sigma$, and H is the unique bounded positive solution to the Riccati equation along the geodesic flow on M.

Y. Mitsumatsu, *A relation between the topological invariance of the Godbillon-Vey invariant and the differentiability of Anosov foliations*, in **Foliations (Tokyo, 1983)**, Adv. Stud. Pure Math., Vol. 5, North-Holland, Amsterdam, 1985, pages 159–167.

Mitsumatsu showed this formula under the assumption that the foliation $\mathcal{F}_g^+$ is C^2. In Section 9 of the paper with Katok we extended this formula to the case when $\mathcal{F}_g^+$ is $C^{1+\alpha}$.

This is fortuitous because of the observation

Corollary. [Mitsumatsu] $\langle GV(\mathcal{F}_g^+), [M] \rangle = 4\pi^2 \cdot \chi(\Sigma)$ if and only if g has constant negative curvature.

Thus, if g a metric with variable negative curvature on Σ, then

$$\langle GV(\mathcal{F}_g^+), [M] \rangle < 4\pi^2 \cdot \chi(\Sigma)$$

The space of metrics with negative curvature on Σ is locally connected, so we obtain for the group cohomology:

Theorem. Let $\mathbf{Diff}^{1+\alpha}(\mathbb{S}^1)$ denote the group of diffeomorphisms of the circle with α-Hölder first derivatives. For $\alpha > 1/2$:

- There is a well-defined class $gv \in H^2(\mathbf{Diff}_\delta^{1+\alpha}(\mathbb{S}^1); \mathbb{R})$.
- There exists continuous families of cycles $f_t \colon \Sigma \to B\mathbf{Diff}_\delta^{1+\alpha}(\mathbb{S}^1)$ such that the evaluation $\langle gv, [f_t\Sigma] \rangle$ varies continuously and non-trivially with t.

Problem 1. Thurston extended his construction of foliations on $\mathbb{S}^3$ with variable Godbillon-Vey class in an unpublished work (see below). Show that the Thurston construction can also be deformed with holonomy in the group $\mathbf{Diff}_{\delta}^{1+\alpha}(\mathbb{S}^1)$.

Problem 2. Heitsch modified Thurston's construction to show that the higher codimension Godbillon-Vey classes are also variable. Show that the Heitsch constructions can also be deformed with holonomy in the group $\mathbf{Diff}_{\delta}^{1+\alpha}(\mathbb{S}^1)$.

⋆ J. Heitsch, *Independent variation of secondary classes*, **Ann. of Math. (2)**, 108:421–460, 1978.

⋆ T. Mizutani, *On Thurston's construction of a surjective homomorphism* $H_{2n+1}(B\Gamma_n, \mathbb{Z}) \to \mathbb{R}$, in **Geometry, dynamics, and foliations 2013**, Math. Soc. Japan, Tokyo, 2017, 211–219.