# Applications of the Contraction Mapping Theorem: Actions on Cantor sets

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- A self-covering map $\phi\colon M \to M$ of a compact manifold without boundary is *expanding* if there exists a Riemannian metric on $TM$ and $\lambda > 1$ such that for all $\vec{v} \in TM$, $\|D\phi(\vec{v})\| \geq \lambda \cdot \|\vec{v}\|$.

**Theorem.** [Franks 1968] Let $M$ be a compact manifold without boundary that admits an expanding map. Then the fundamental group $\Gamma = \pi_1(M, x)$ has polynomial growth, and hence contains a nilpotent subgroup of finite index.


⋆ Michael Shub, *Expanding maps*, **Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968)**, Amer. Math. Soc., Providence, RI, 1970, pages 273–276.

⋆ M. Gromov, *Groups of polynomial growth and expanding maps*, **Inst. Hautes Études Sci. Publ. Math.**, 53:53–73, 1981.

Let $\phi\colon M \to M$ be an expanding map with a fixed point $x \in M$.

Set $\Gamma = \pi_1(M, x)$, then get embedding $\varphi = \phi_\#\colon \Gamma \to \Gamma$.

Set $\Gamma_0 = \Gamma$ and $\Gamma_{\ell+1} = \varphi(\Gamma_\ell)$ for all $\ell \geq 0$.

As $\phi$ is expanding, we have $\cap_{\ell>0} \Gamma_\ell \ = \ \{0\}$.

**Definition:** Let $\Gamma$ be a finitely generated group. A proper inclusion $\varphi\colon \Gamma \to \Gamma$ with finite index image is called a renormalization of $\Gamma$.

The terminology is motivated by the case when $\Gamma = \mathbb{Z}^n$.

**Example.** $\Gamma = \mathbb{Z}^2$ and choose $p, q > 1$ integers. Then $\varphi(a, b) = (pa, qb)$ is a renormalization.

**Definition:** $\Gamma$ is *non co-Hopfian* if it admits a renormalization.

Benjamini, and Nekrashevych and Pete introduced the terminology

**Definition:** $\Gamma$ is *strongly scale invariant* if there is a renormalization $\varphi\colon \Gamma \to \Gamma$ such that, for $\Gamma_\ell = \varphi^\ell(\Gamma)$,

$$K(\varphi) = \bigcap_{\ell > 0} \Gamma_\ell \quad \text{is a finite group}$$

**Conjecture:** Strongly scale-invariant group $\Gamma$ is virtually nilpotent.

$\star$ V. Nekrashevych and G. Pete, *Scale-invariant groups*, **Groups Geom. Dyn.** 5:139–167, 2011.

Franks-Gromov theorem shows that the conjecture is true if $\Gamma$ is the fundamental group of a compact manifold with an expanding map.

Another special case of the conjecture is known:

**Theorem:** Let $\Gamma$ be a strongly scale-invariant group, with a renormalization $\varphi\colon \Gamma \to \Gamma$ such that $\Gamma_\ell = \varphi^\ell(\Gamma)$ is normal in $\Gamma$. Then $\Gamma/K(\varphi)$ is abelian.

$\star$ *W. Van Limbeek*, **Towers of regular self-covers and linear endomorphisms of tori**, Geom. Topol., 2018.

It is not usual that the subgroups $\Gamma_\ell$ are normal in $\Gamma$.

Let $\mathcal{H}$ denote the 3-dimensional Heisenberg group represented as $(\mathbb{Z}^3, *)$ with the group operation $*$, so for $x, u, y, v, z, w \in \mathbb{Z}$,

$$(x, y, z) * (u, v, w) = (x + u, y + v, z + w + xv)$$

$$(x, y, z)^{-1} = (-x, -y, -z + xy)$$

This is equivalent to the upper triangular representation in $GL(\mathbb{Z}^3)$. In particular,

$$(x, y, z) * (u, v, w) * (x, y, z)^{-1} = (u, v, w + xv - yu)$$

For integers $p, q > 1$ let $\varphi \colon \mathcal{H} \to \mathcal{H}$ by $\varphi(x, y, z) = (px, qy, pqz)$.

$$\mathcal{H}_\ell = \varphi^\ell(\mathcal{H}) = \{(p^\ell x, q^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}$$

$$\bigcap_{\ell > 0} \mathcal{H}_\ell = \{e\}$$

Assume that $p, q > 1$ are distinct prime numbers.

The *normal core* for $\mathcal{H}_\ell$ is the largest normal subgroup of $\mathcal{H}_\ell$. Then

$$C_\ell = \mathrm{core}(\mathcal{H}_\ell) = \{((pq)^\ell x, (pq)^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}$$

is a proper subgroup of $\mathcal{H}_\ell$, with quotient group

$$\mathcal{H}_\ell / C_\ell \cong \{(x, y, 0) \mid x \in \mathbb{Z}/q^\ell\mathbb{Z} \ , \ y \in \mathbb{Z}/p^\ell\mathbb{Z}\}$$

So as $\ell \to \infty$, the groups $\Gamma_\ell$ become less and less normal.

Let $\Gamma$ be a finitely generated group.

**Definition.** A *group chain* is a descending chain $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$ where $\Gamma_0 = \Gamma$ and $\Gamma_{\ell+1} \subset \Gamma_\ell$ is a proper subgroup of finite index.

The quotient $X_\ell = \Gamma/\Gamma_\ell$ is a finite set with transitive left $\Gamma$-action.

Inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a surjection $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Define

$$\mathfrak{X} \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \ .$$

The product of finite sets is given the Tychonoff topology - cylinder sets generate the topology.

Then $\mathfrak{X}$ is a closed subset, so is a Cantor space with left $\Gamma$-action.

A point in $\mathfrak{X}$ is a sequence $x = (x_0, x_1, \ldots)$ where $p_{\ell+1}(x_{\ell+1}) = x_\ell$.

For $\gamma \in \Gamma$ we have $\Phi(\gamma)(x) = \gamma \cdot x = (\gamma \cdot x_0, \gamma \cdot x_1, \ldots)$.

$\mathfrak{X}$ has an ultrametric $d_{\mathfrak{X}}$:

for $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$

$$d_{\mathfrak{X}}(x, y) = 2^{-m} \quad \text{where} \quad m = \min\{\ell \mid x_\ell \neq y_\ell\}$$

This satisfies the *ultrametric* triangle inequality, for $x, y, z \in \mathfrak{X}$,

$$d_{\mathfrak{X}}(x, y) \leq \min\{d_{\mathfrak{X}}(x, z), d_{\mathfrak{X}}(z, y)\}$$

Then $d_{\mathfrak{X}}(\gamma \cdot x, \gamma \cdot y) = d_{\mathfrak{X}}(x, y)$ so $d_{\mathfrak{X}}$ is Γ-invariant ultrametric.

The action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is a *generalized odometer*.

⋆ M.-I. Cortez and S. Petite, *G-odometers and their almost one-to-one extensions*, **J. London Math. Soc.**, 78(2):1–20, 2008.

A Cantor action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is *equicontinuous* if for some metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\mathfrak{X}}(x,y) < \delta \implies d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \epsilon \quad \text{for all } g \in \Gamma.$$

The action $\Phi$ is isometric for the ultrametric the $d_{\mathfrak{X}}$ on $\mathfrak{X}$, so

- $(\mathfrak{X}, \Gamma, \Phi)$ is an *equicontinuous Cantor action*.

**Remark:** A minimal equicontinuous Cantor action can also be viewed as a group action on a rooted tree, aka an *arboreal action*.

$\star$ R.I. Grigorchuk, *Some problems of the dynamics of group actions on rooted trees*, **Proc. Steklov Institute of Math.**, 273: 64–175, 2011.

Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous Cantor action.

The action is a homomorphism $\Phi \colon \Gamma \to \mathbf{Homeo}(\mathfrak{X})$, and let

$\widehat{\Gamma} = \overline{\Phi(\Gamma)} \subset \mathbf{Homeo}(\mathfrak{X})$ denote the closure in uniform topology

**Proposition:** [Ellis, 1969] $\Phi$ equicontinuous implies that $\widehat{\Gamma}$ is a profinite group, that is, *compact* and t*otally disconnected*.

**Lemma:** $\Phi$ minimal action implies that $\widehat{\Gamma}$ acts transitively on $\mathfrak{X}$.

For $x \in \mathfrak{X}$, define the isotropy subgroup

$$\mathcal{D}_x = \{\widehat{\gamma} \in \widehat{\Gamma} \mid \widehat{\Phi}(\widehat{\gamma})(x) = x\}$$

The isomorphism class of $\mathcal{D}_x$ is independent of choice of $x$, and $\mathfrak{X} \cong \widehat{\Gamma}/\mathcal{D}_x$ as left $\Gamma$-spaces.

$\widehat{\Gamma}$ has a discrete description analogous to that for $\mathfrak{X}$.

The action of $\Gamma$ on $X_\ell = \Gamma/\Gamma_\ell$ permutes the cosets, where $\Gamma_\ell$ is the isotropy of the basepoint $e \cdot \Gamma_\ell$.

The kernel of the action $\Gamma \to \mathbf{Perm}(X_\ell)$ is the core normal subgroup of $\Gamma_\ell$,

$$C_\ell = \bigcap_{\gamma \in \Gamma} \gamma \cdot \Gamma_\ell \cdot \gamma^{-1}$$

Then there is an isomorphism

$$\widehat{\Gamma} = \varprojlim \{\widehat{p}_{\ell+1} \colon \Gamma/C_{\ell+1} \to \Gamma/C_\ell \mid \ell \geq 0\}$$

and the isotropy subgroup is

$$\mathcal{D}_x = \varprojlim \{\widehat{p}_{\ell+1} \colon \Gamma_{\ell+1}/C_{\ell+1} \to \Gamma_\ell/C_\ell \mid \ell \geq 0\}$$

Now consider a renormalization $\varphi\colon \Gamma \to \Gamma$ and define $\Gamma_\ell = \varphi^\ell(\Gamma)$ with associated inverse limit space $\mathfrak{X}_\varphi = \widehat{\Gamma}_\varphi / \mathcal{D}_\varphi$ and $\Gamma$-action $\Phi_\varphi$.

**Proposition.** $\varphi$ induces a contraction $\lambda_\varphi\colon \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$.

*Proof.* $\varphi$ induces a map of quotients $\overline{p}\colon \Gamma/\Gamma_\ell \to \Gamma_1/\Gamma_{\ell+1}$. This induces the right shift map $\lambda_\varphi\colon \mathfrak{X}_\varphi \to U_1 \subset \mathfrak{X}_\varphi$,

$$\lambda_\varphi(x_0, x_1, \ldots) = (e, \varphi(x_0), \varphi(x_1), \ldots)$$

The contraction has a unique fixed-point $x_\varphi \in \mathfrak{X}_\varphi$.

How does this help with showing that $\Gamma$ is virtually nilpotent?

What we need is a contraction $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

We take a detour into Cantor dynamics.

Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be a Cantor action of a countable group $\Gamma$.

The topology of Cantor space $\mathfrak{X}$ is generated by clopen subsets: U is closed and open.

A non-empty clopen $U \subset \mathfrak{X}$ is adapted if the return times to $U$ form a subgroup:

$$\Gamma_U = \{g \in \Gamma \mid \Phi(g)(U) = U\} \subset \Gamma$$

**Lemma:** Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuos Cantor action. For $x \in \mathfrak{X}$ and $V$ open with $x \in V$, there is an adapted set $U$ with $x \in U \subset V$.

**Definition:** A Cantor action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$ is quasi-analytic if for each adapted clopen set $U \subset \mathfrak{X}$ and $g \in H$ we have

- if $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on $U$, then $\Phi(g)$ acts as the identity on all of $\mathfrak{X}$.

For $H$ a countable group, this is equivalent to topologically free.

Here is our key technical result:

**Theorem 1.** The action $\widehat{\Phi}_\varphi\colon \widehat{\Gamma}_\varphi \times \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$ is quasi-analytic.

**Corollary.** Let $\widehat{\gamma} \in \widehat{\Gamma}_\varphi$. The homeomorphism $\widehat{\Phi}_\varphi(\widehat{\gamma})\colon \mathfrak{X}_\varphi \to \mathfrak{X}_\varphi$ is uniquely determined by its restriction to an adapted subset of $\mathfrak{X}$.

Theorem 1 is the key to the proof of the next two results:

**Theorem 2.** A renormalization map $\varphi$ induces a contraction map on the closure, $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ with open image.

**Theorem 3.** $\mathcal{D}_\varphi = \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi$

This connects discriminant invariants for Cantor actions, with invariants for contraction profinite groups.

For the proof see

⋆ S. Hurder, O. Lukina and W. Van Limbeek, *Cantor dynamics of renormalizable groups*, **Groups, Geometry, and Dynamics**, 15:1449-1487, 2021.

The proof of Theorem 2 has a key point.

The renormalization $\varphi\colon \Gamma \to \Gamma$ naturally induces a map $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \textbf{Homeo}(U_1)$ where $U = \mathrm{image}(\lambda_\varphi) \subset \mathfrak{X}_\varphi$.

We need to show that the maps in the image of $\widehat{\varphi}$ have unique extensions to $\textbf{Homeo}(\mathfrak{X}_\varphi)$.

This is exactly what Theorem 1 says is true.

Let $\mathcal{H}$ denote the 3-dimensional Heisenberg group. Assume that $p, q > 1$ are distinct prime numbers, and define $\varphi \colon \mathcal{H} \to \mathcal{H}$ by $\varphi(x, y, z) = (px, qy, pqz)$. Then

$$\mathcal{H}_\ell = \varphi^\ell(\mathcal{H}) = \{(p^\ell x, q^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}$$

$$C_\ell = \mathrm{core}(\mathcal{H}_\ell) = \{((pq)^\ell x, (pq)^\ell y, (pq)^\ell z) \mid x, y, z \in \mathbb{Z}\}$$

$$\mathcal{H}_\ell / C_\ell \cong \{(x, y, 0) \mid x \in \mathbb{Z}/q^\ell\mathbb{Z} \ , \ y \in \mathbb{Z}/p^\ell\mathbb{Z}\}$$

Then taking the inverse limit of $\widehat{p}_{\ell+1} \colon \Gamma / C_{\ell+1} \to \Gamma / C_\ell$

$$\widehat{\Gamma} \cong \{(a, b, c) \mid a, b, c \in \widehat{\mathbb{Z}}_{pq}\}$$

and the inverse limit of $\widehat{p}_{\ell+1} \colon \Gamma_{\ell+1}/C_{\ell+1} \to \Gamma_\ell / C_\ell$

$$\mathcal{D}_\varphi = \{(a, b, 0) \mid a \in \widehat{\mathbb{Z}}_q, b \in \widehat{\mathbb{Z}}_p\} \cong \widehat{\mathbb{Z}}_q \times \widehat{\mathbb{Z}}_p$$

$\varphi$ maps $\mathcal{D}_\varphi$ to itself and is multiplication by $p$ and $q$, respectively.

Theorem 2 gives us a contraction mapping $\widehat{\varphi}\colon \widehat{\Gamma}_{\varphi} \to \widehat{\Gamma}_{\varphi}$ with open image, where $\widehat{\Gamma}_{\varphi}$ is a Cantor group, so is a complete metric space.

There is an extensive literature on the structure of profinite groups with an open contraction mapping. We use results from:


⋆ U. Baumgartner and G. Willis, *Contraction groups and scales of automorphisms of totally disconnected locally compact groups*, **Israel J. Math.**, 142:221–248, 2004.

⋆ C. Reid, *Endomorphisms of profinite groups*, **Groups Geom. Dyn.**, 8:553–564, 2014.

**Theorem.** Let $\widehat{\varphi}\colon \widehat{\Gamma}_\varphi \to \widehat{\Gamma}_\varphi$ be a contraction map with open image. Then there is an isomorphism with a semi-direct product

$$\widehat{\Gamma}_\varphi \cong \mathcal{N}_\varphi \rtimes \mathcal{D}_\varphi$$

$$\begin{aligned} \mathcal{N}_\varphi &= \{\widehat{g} \in \widehat{\Gamma}_\varphi \mid \lim_{\ell\to\infty} \widehat{\varphi}^\ell(\widehat{g}) = \widehat{e}\} \\ \mathcal{D}_\varphi &= \bigcap_{n>0} \widehat{\varphi}^n(\widehat{\Gamma}_\varphi) \subset \widehat{\Gamma}_\varphi \end{aligned}$$

Moreover, the contraction factor $\mathcal{N}_\varphi$ is pro-nilpotent.

That is, $\mathcal{N}_\varphi$ is an inverse limit of nilpotent groups.

We use this structure theorem for contraction maps to show:

**Theorem.** Let $\varphi$ be a renormalization of the finitely generated group $\Gamma$. Suppose that

$$K(\varphi) = \bigcap_{\ell > 0} \varphi^{\ell}(\Gamma) \subset \Gamma \quad , \quad \mathcal{D}_{\varphi} = \bigcap_{n>0} \widehat{\varphi}_0^n(\widehat{\Gamma}_{\varphi}) \subset \widehat{\Gamma}_{\varphi}$$

are both finite groups, then

- $\Gamma$ is virtually nilpotent,
- If both groups are trivial, then $\Gamma$ is nilpotent.

**Remark:** The normality assumption in van Limbeek's Theorem is replaced by the assumption that $\mathcal{D}_{\varphi}$ is a finite group.

**Basic Problem:** Show that $\mathcal{D}_{\varphi}$ is virtually nilpotent.

The Conjecture has been shown for another special case:

**Theorem.** Let $\Gamma$ be a finitely generated strongly scale invariant group, and assume that $\Gamma$ is virtually polycyclic. Then $\Gamma$ is virtually nilpotent.

⋆ J. Deré, *Strongly scale-invariant virtually polycyclic groups*, **Groups Geom. Dyn.**, 16:985–1004, 2022.

Let $\Gamma$ be a residually finite finitely generated group.

Then $\Gamma$ has at least one group chain $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$.

We then obtain an action $\widehat{\Phi} \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$, where $\mathfrak{X} \cong \widehat{\Gamma}/\mathcal{D}_x$.

**Problem.** How are the dynamical properties of this Cantor action related to the group structure of $\Gamma$?

One motivation for the problem is the observation that associated to the action $(\mathfrak{X}, \Gamma, \Phi)$ is a cross-product $C^*$-algebra $C^*(\mathfrak{X}, \Gamma, \Phi)$. This construction of $C^*$-algebras appears in many contexts.

**Problem.** How is the structure of $C^*(\mathfrak{X}, \Gamma, \Phi)$ related to the dynamical properties of the Cantor action $(\mathfrak{X}, \Gamma, \Phi)$?

## Cantor action bibliography.


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⋆ O. Lukina, *Arboreal Cantor actions*, **J. Lond. Math. Soc. (2)**, 99:678–706, 2019.

⋆ S. Hurder and O. Lukina, *Orbit equivalence and classification of weak solenoids*, **Indiana Univ. Math. J.**, 69:2339–2363, 2020.

⋆ O. Lukina, *Galois Groups and Cantor actions*, **Transactions A.M.S.**, 374:1579–1621, 2021.

⋆ M. Gröger and O. Lukina, *Measures and regularity of group Cantor actions*, **Discrete Contin. Dyn. Syst. Ser. A.**, 41:2001–2029, 2021.

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