Classifying Foliations

Steven Hurder

University of Illinois at Chicago
www.math.uic.edu/∼hurder

"Foliations, Topology and Geometry in Rio"
On the occasion of the 70th birthday of Paul Schweitzer

1981–1983: Institute for Advanced Study

In Spring 1982, news arrived at the IAS of Gérard Duminy's breakthrough:

THEOREM: [Duminy] Let $\mathcal{F}$ be a C^2-foliation of codimension one on a compact manifold M. If the Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M)$ is non-trivial, then $\mathcal{F}$ has a resilient leaf, and hence an uncountable set of leaves with exponential growth.

In a seminar that Spring at the IAS, including Paul, Larry Conlon, James Heitsch, the speaker and others, Duminy's hand-written manuscript with the proof was presented and dissected.

This seed inspired 25 years of subsequent work.

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Question 3': Is it possible to classify (almost all) foliations on M based on their dynamical behavior?

Foliation dynamics

- A continuous dynamical system on a compact manifold M is a flow $\varphi\colon M \times \mathbb{R} \to M$, where the orbit $L_x = \{\varphi_t(x) = \varphi(x, t) \mid t \in \mathbb{R}\}$ is thought of as the time trajectory of the point $x \in M$. The trajectories of the points of M are necessarily points, circles or lines immersed in M, and the study of their aggregate and statistical behavior is the subject of ergodic theory for flows.

- In foliation dynamics, we replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the dynamics of $\mathcal{F}$ asks for properties of the aggregate and statistical behavior of the collection of its leaves.

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Theorem: (Haefliger [1970]) Each C^r-foliation $\mathcal{F}$ on M of codimension q determines a well-defined map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r$ whose homotopy class in uniquely defined by $\mathcal{F}$.

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Theorem: (Thurston [1975]) Each "natural" map $h_{\mathcal{F}} \colon M \to B\Gamma_q^r \times BO_p$ yields a C^r-foliation $\mathcal{F}$ on M with concordance class determined by $h_{\mathcal{F}}$.

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The topological type of $B\Gamma_q^r$ is analyzed using the "linearization" of the normal structure along the leaves – the Bott connection and its invariants.

Secondary classes

Assume $\mathcal{F}$ is C^r-foliation with $r \geq 2$.

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Theorem: (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972]) For each codimension q, there is a non-trivial space of secondary invariants $H^*(WO_q)$ and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{array}{ccc} & \overset{\tilde{\Delta}}{\nearrow} & H^*(B\Gamma_q; \mathbb{R}) \\ & & \downarrow h_{\mathcal{F}}^* \\ H^*(WO_q) & \xrightarrow{\Delta} & H^*(M; \mathbb{R}) \end{array}$$

The study of these maps has been the principle source of information about the (non-trivial) homotopy type of $B\Gamma_q^r$ for $r \geq 2$.

Homotopy chaos

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Secondary classes measure some uncountable aspect of foliation geometry.

$C^{1+\alpha}$ is essential

In contrast, Takashi Tsuboi proved the following amazing result:

Theorem: (Tsuboi [1989]) The classifying map of the normal bundle $\nu\colon B\Gamma_q^1 \to BO(q)$ is a homotopy equivalence.

The proof is a technical tour-de-force, using Mather-Thurston type techniques for the study of $B\Gamma$, along with (to paraphrase) "smearing along orbits in acyclic models".

Ergodic theory & secondary classes

Mizutani, Morita and Tsuboi [1981], Duminy & Sergiescu [1981], and Duminy [1982] developed techniques of localizing the Godbillon-Vey class to open saturated subsets, first for foliations of depth 1, and then for arbitrary depth. Heitsch & Hurder [1984] extended the localization technique to saturated measurable subsets. Then add two key ideas:

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Idea 1: (Heitsch & Hurder) The normal derivative cocycle used to define the forms $\Delta(h_I)$ appearing in the secondary class $\Delta(h_I c_J)$ is only required to be smooth along leaves, and measurable transversally. Thus, the contribution of $\Delta(h_I)$ can be estimated using ergodic theory techniques for the measurable equivalence relation defined by $\mathcal{F}$.

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Idea 2: (Hurder & Katok) Before passing to cohomology or homotopy, "smear along orbits the linearization data" for $B\Gamma_q^r$. More precisely, use the ergodic theory and dynamical data for the foliation to "optimally temper" the normal derivative cocycle.

Pseudogroups & Groupoids

Every foliation admits a discrete model by choosing a section $\mathcal{T} \subset M$, an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$.

Definition: A pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if there is

- a relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}$ such that $\langle \Gamma \rangle = \mathcal{G}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

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Definition: The groupoid of $\mathcal{G}$ is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \ \& \ x \in D(g)\} \ , \ \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $\quad T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q, \quad T_x\mathcal{T} \cong_x \mathbb{R}^q \quad$ for all $x \in \mathcal{T}$.

Definition: The normal cocycle $D\varphi \colon \Gamma_{\mathcal{G}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D\varphi[g]_x = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D[h]_y \cdot D[g]_x$$

Pseudogroup word length

Definition: For $g \in \Gamma_{\mathcal{G}}$, the word length $\|[g]\|_x$ of the germ $[g]_x$ of g at x is the least n such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_n}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another.

Asymptotic exponent

Definition: The transverse expansion rate function at x is

$$\lambda(\mathcal{G}, n, x) = \max_{\|[g]\|_x \leq n} \frac{\ln\left(\max\{\|D_x g\|, \|D_y g^{-1}\|\}\right)}{\|[g]\|_x} \;\geq\; 0$$

Definition: The asymptotic transverse growth rate at x is

$$\lambda(\mathcal{G}, x) = \limsup_{n\to\infty} \; \lambda(\mathcal{G}, n, x) \;\geq\; 0$$

This is essentially the maximum Lyapunov exponent for $\mathcal{G}$ at x.

Expansion classification

$$M = \mathcal{E} \cup \mathcal{P} \cup \mathcal{H}$$

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2. Parabolic points: $\mathcal{P} \cap \mathcal{T} = \{x \in \mathcal{T} - (\mathcal{E} \cap \mathcal{T}) \mid \lambda(\mathcal{G}, x) = 0\}$
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3. Partially Hyperbolic points: $\mathcal{H} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}, x) > 0\}$
   i.e., "points of exponential-growth expansion" or non-uniformly, partially hyperbolic transverse dynamics

$\mathcal{E} \sim$ measurable Riemannian structure

Theorem: There exists a measurable Riemannian metric on the normal bundle $Q \mid \mathcal{E}$ which is holonomy invariant.

Proof. Introduce the space of fiberwise Riemannian metrics $\mathcal{S} = GL(Q)/O(q) \to M$ on which the derivative cocycle $D\varphi$ acts isometrically on the fiberwise symmetric spaces $GL(Q_x)/O(q)$.

A measurable section $\sigma \colon \mathcal{E} \to \mathcal{S}$ corresponds to a measurable transverse metric on $\mathcal{E}$, and the action of $D\varphi$ extends to an action on such sections. Let σ_0 be a smooth metric on Q restricted to $\mathcal{E}$.

For $x \in \mathcal{E}$ there is an upper bound on the distance between $\sigma_0(x)$ and $[g]_x \cdot \sigma_0(x)$ for all $[g]_x \in \Gamma_{\mathcal{G}}$. Hence we can use a center of mass construction to obtain a section σ which is invariant. □

Examples with $M = \mathcal{E}$

Example: $\mathcal{F}$ is Riemannian $\Rightarrow M = \mathcal{E}$.

Is this the only example?

Question: If $M = \mathcal{E}$, must $\mathcal{F}$ be Riemannian?

In the case where $\mathcal{F}$ is defined by a smooth measure-preserving action of a higher rank lattice Γ on a compact manifold, this is a well-known (old) question of Robert Zimmer, which has recently been shown true by David Fisher and Gregory Margulis if Γ has Property T.

$\mathcal{P} \sim$ almost invariant metric

Theorem: For all $\epsilon > 0$, there exists a measurable Riemannian metric σ_ϵ on the normal bundle $Q \mid \mathcal{P}$ which is ϵ-holonomy invariant.

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Proof. Much the same as above, but using tempering of cocycles and techniques from the papers:

• [Hurder & Katok 1987] "Ergodic theory and Weil measures for foliations", Ann. of Math. (2) 126 (1987)

• [Hurder & Langevin 2004] "Dynamics and the Godbillon-Vey Class of C^1 Foliations", Jour. Diff. Geometry (to appear)

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Example: A foliation $\mathcal{F}$ is distal if its pseudogroup $(\mathcal{G}_{\mathcal{F}}, \mathcal{T})$ is distal: that is, for all $x \neq y \in \mathcal{T}$ there exists $\epsilon_{x,y} > 0$ such that

$$d_{\mathcal{T}}(g(x), g(y)) \geq \epsilon_{x,y} \text{ for all } g \in \mathcal{G}_{\mathcal{F}}$$

For example, all compact foliations are distal.

Theorem: If $\mathcal{F}$ is distal and $\mathbf{C}^{1+\alpha}$ for some $\alpha > 0$, then $M = \mathcal{P}$.

Examples with $M = \mathcal{P}$

Theorem: (Clark & Hurder [2006]) Suppose that $\mathcal{F}$ has a compact leaf L with $H^1(L, \mathbb{R}) \neq 0$, and there is a saturated open neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation. Then there is an arbitrarily small smooth perturbation $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}'$ has a solenoidal minimal set $\mathbf{K} \subset U$, where the leaves of $\mathcal{F}' \mid \mathbf{K}$ all cover L. Moreover, if $\mathcal{F}$ is distal, then $\mathcal{F}'$ is distal.

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Problem: If $M = \mathcal{P}$, does there exists a structure theory for the minimal sets of $\mathcal{F}$? For example, must such $\mathbf{K}$ admit a topological Lie group structure transversally, or have a factor with this property?

Secondary classes and dynamics

Recall that a secondary class $y_I c_J \in H^*(WO_q)$ is *residual* if c_J has degree $2q$. The two results above then imply:

Theorem: (Hurder) If $y_I c_J \in H^*(WO_q)$ is a residual secondary class (e.g., Godbillon-Vey type) then the localizations $\Delta(y_I c_J)|\mathcal{E} = 0$ and $\Delta(y_I c_J)|\mathcal{P} = 0$. Hence, if $\Delta(y_I c_J)$ non-zero implies that $\mathcal{H}$ has positive Lebesgue measure.

Thus, understanding the dynamical meaning of the residual secondary classes requires understanding the dynamics of foliations which have non-uniformly, partially hyperbolic behavior on a set of positive measure.

$\mathcal{H} \sim$ codimension one

Theorem: [Hurder (2005)] Let $\mathcal{G}$ be a compactly generated $C^{1+\alpha}$-pseudogroup, where the Hölder exponent $\alpha > 0$, and $\mathcal{T}$ has dimension one. The for every minimal set $\mathbf{K} \subset \mathcal{T}$ the intersection $\mathbf{K} \cap \mathcal{H}$ has Lebesgue measure zero.

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Combining this with results from Poincaré-Bendixsion Theory for C^2-foliations, one gets:

Theorem: [Hurder 2005] Let $\mathcal{F}$ be a C^2-foliation of codimension $q = 1$ with $GV(\mathcal{F}) \neq 0$. Then there is an open subset $U \subset M$ with

- U is saturated by the leaves of $\mathcal{F}$,
- U contains the support of the cohomology class $GV(\mathcal{F})$
- U contains a dense collection of chaotic laminations.
- $\mathcal{F}|U$ is expansive

Geometric entropy

Given a subset $X \subset \mathcal{T}$, $\mathcal{S} = \{x_1, \ldots, x_\ell\} \subset X$ is (n, ϵ)-separated if

$$\forall\ x_i \neq x_j\ ,\ \exists\ g \in \mathcal{G}|X \ \text{ such that } \ \|g\|_{x_i} \leq n \ \&\ d_{\mathcal{T}}(g(x_i), g(x_j)) \geq \epsilon$$

Then set

$$h(X, n, \epsilon) = \max\ \#\{\mathcal{S} \mid \mathcal{S} \subset X \text{ is } (n, \epsilon) \text{ separated}\}$$

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Definition: (Ghys, Langevin, Walczak [1986])

$$h(\mathcal{G}) = \lim_{\epsilon \to 0} \left\{ \limsup_{n \to \infty} \frac{\ln h(\mathcal{T}, n, \epsilon)}{n} \right\}$$

The geometric entropy of $\mathcal{F}$ is $h(\mathcal{F}) = h(\mathcal{G}_{\mathcal{F}})$.

Proposition: If $\mathcal{G}$ contains a Markov subpseudogroup, then $h(\mathcal{G}) > 0$.

Local entropy

Local entropy for measure-preserving transformations was introduced by Brin & Katok at a talk in Rio de Janeiro in 1981. There is a very useful version of this notion for pseudogroups.

Let $B(x, \delta) \subset \mathcal{T}$ denote the open δ–ball about $x \in \mathcal{T}$.

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Proposition: (Hurder [2005]) $\mathcal{G}$ a finitely-generated pseudogroup:

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Entropy and chaos and $\mathcal{H}$

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No Problem: Happy Birthday, Paul!!