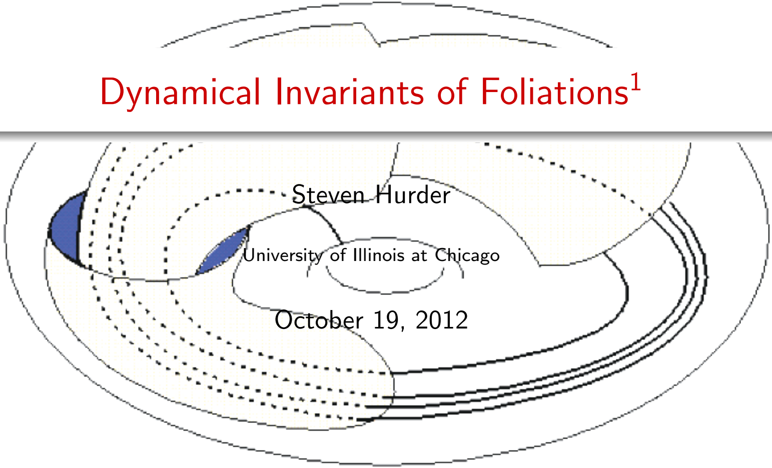
Dynamical Invariants of Foliations[1]



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October 19, 2012

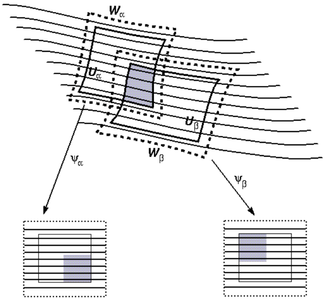
[1] Colloquium GT3, IRMA, Université de Strasbourg

Foliations

A foliation $\mathcal{F}$ of a compact manifold M is . . .

• a "uniform partition" of M into submanifolds of constant dimension p and codimension q.

Definition: M a smooth manifold of dimension n is *foliated* if there is a covering of M by coordinate charts whose change of coordinate functions preserve leaves



Reeb, Vincensini & Ehresmann (1949)

André Haefliger, *Naissance des feuilletages, d'Ehresmann-Reeb à Novikov*, **Géométrie au vingtième siècle : 1930-2000**, Hermann, Éditeurs des Sciences et des Arts, Paris. http://www.unige.ch/math/folks/haefliger/Feuilletages.pdf

Foliations

A foliation $\mathcal{F}$ of a compact manifold M is also ...

• a local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle for a "good covering". (Ehresmann, Haefliger)

• a dynamical system on M with multi-dimensional time.

• a groupoid $\Gamma_{\mathcal{F}} \to M$ with fibers complete manifolds, the holonomy covers of leaves.

Each point of view has advantages and disadvantages.

Problem: How to distinguish and classify foliations?

Foliation invariants

Each viewpoint yields its own classes of invariants to distinguish foliations.

• a “uniform partition” of M into submanifolds of constant dimension p

$\Longrightarrow$ Growth rates of leaves, other quasi-isometry invariants.

• local geometric structure on M, given by a $\Gamma_{\mathbb{R}^q}$-cocycle

$\Longrightarrow$ Secondary classes, other cohomology invariants derived from sheaves.

• dynamical system on M with multi-dimensional time

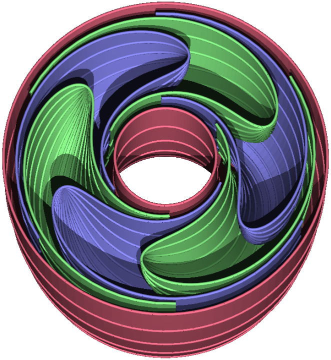
$\Longrightarrow$ Geometric entropy, Lyapunov spectrum, invariant & harmonic measures.

• groupoid $\Gamma_{\mathcal{F}} \to M$ with fibers the holonomy covers of leaves

$\Longrightarrow$ C^*-algebras $C^*_r(M, \mathcal{F})$ and its K-Theory invariants, von Neumann algebra $W^*(M, \mathcal{F})$ and its flow of weights.

Reeb foliation

Here is maybe the prettiest picture of the Reeb foliation of $\mathbb{S}^3$, from the Wikimedia Commons:



• It's Godbillon-Vey class is zero.

Mizutani, Morita & Tsuboi: *The Godbillon-Vey classes of codimension one foliations which are almost without holonomy*, [Annals of Math., 1981]

This was the model for studies of how the Godbillon-Vey class is related to dynamics, culminating in the celebrated result of Gerard Duminy in 1982 (discussed later).

• The K-Theory of its C^*-algebra can be explicitly calculated:

Anne-Marie Torpe: *K-theory for the leaf space of foliations by Reeb components*, [J. Funct. Anal., 1985]

This was a model example for the geometric approach to foliation index theory:

Paul Baum & Alain Connes: *Geometric K-theory for Lie groups and foliations*, [Enseign. Math., 2000: preprint, 1985]

Not bad, for such a "modest" example. The status of the classification problem today - 30 years later - has seen great progress.

Riemannian foliations

$\mathcal{F}$ is a *Riemannian foliation* if there is a Riemannian metric on TM so that its restriction to $Q = TM/T\mathcal{F}$ is invariant under the leafwise parallelism.

Theorem: [Molino, 1984] Let $\mathcal{F}$ be a smooth Riemannian foliation on a compact manifold M. Then for each leaf L, its closure $\overline{L}$ is a manifold, and:

1. the restricted foliation $\mathcal{F} \mid \overline{L}$ is Riemannian, even homogeneous;
2. all leaves of $\mathcal{F}$ in $\overline{L}$ are dense in $\overline{L}$;
3. the closures of the leaves form a singular Riemannian foliation of M.

If the group of foliated homeomorphisms of M is *transitive*, then the foliation by leaf closures is defined by a submersion to the quotient manifold $W = M/\overline{T\mathcal{F}}$.

Remark: Molino's Theorem and related works by Carrière, Ghys, and others give (almost) a complete classification of Riemannian foliations in low dimensions.

Anosov foliations

Anosov foliations are at the opposite extreme from Riemannian foliations:

$$TM = E^{+} \oplus \langle \vec{X} \rangle \oplus E^{-}$$

where the flow φ_t of $\vec{X}$ uniformly expands E^{+}, and uniformly contracts E^{-}.

$\mathcal{F}^{\pm}$ is the foliation given by the integral manifolds of the distribution $\vec{X} \oplus E^{\pm}$.

The restriction of a Riemannian metric on TM to $Q^{\pm} \cong E^{\mp}$ is either uniformly contracted/expanded under the leafwise parallelism along $\vec{X}$.

Anosov foliations are extremely well-studied, and though not classified, there are algebraic models for "what they should be", and proofs that they are algebraic in important cases.

Concept extends to actions of Lie groups, and suspensions of countable groups acting smoothly on compact manifolds, where there are many open questions.

Intermediate classes of foliation dynamics

The two cases above have exceptional uniformity in their structure:

• For a Riemannian foliation $\mathcal{F}$, all leaves have quasi-isometric & diffeomorphic holonomy coverings, given by a "typical" leaf $L \subset M$.

• For an Anosov foliation $\mathcal{F}$, all leaves are quasi-isometric & diffeomorphic, to either $\mathbb{R}^n$ or to a simply-connected nil-manifold $\mathcal{N}$.

For the general foliation $\mathcal{F}$, the closure of a leaf L can contain a variety of diffeomorphism types of manifolds as leaves (Reeb foliation, and beyond).

• Foliation dynamics is a "mash-up" of these two extremes of behavior.

• Foliation dynamics defined "germinally", as no uniform notion of "time".

The problem in the study of foliation theory:

"How to obtain global conclusions from local assumptions?"

Pseudogroups

A section $\mathcal{T} \subset M$ for $\mathcal{F}$ is an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ on $\mathcal{T}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$.

Definition: A pseudogroup of transformations $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{T}$ is *compactly generated* if there is

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all orbits of $\mathcal{G}_{\mathcal{F}}$;
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ which generates $\mathcal{G}_{\mathcal{F}}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Groupoid word length

Definition: The groupoid of $\mathcal{G}_{\mathcal{F}}$ is the space of germs

$$\Gamma_{\mathcal{F}} = \{[g]_x \mid g \in \mathcal{G}_{\mathcal{F}} \ \& \ x \in D(g)\} \ , \ \Gamma_{\mathcal{F}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

The *word length* $\|[g]\|_x$ of the germ $[g]_x$ of g at x is the least k such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_k}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another along an orbit or leaf, while preserving the germinal dynamics.

Derivative cocycle

There is really no good ideas for classifying foliations which are just C^0.

Assume that the holonomy maps in $\mathcal{G}_{\mathcal{F}}$ are uniformly C^1.

Assume $(\mathcal{G}_{\mathcal{F}}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q, \quad T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: The *normal derivative cocycle* $D \colon \Gamma_{\mathcal{F}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D([g]_x) = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D([h]_y \circ [g]_x) = D([h]_y) \cdot D([g]_x)$$

Problem: Extract from the normal derivative cocycle classification data: dynamical invariants for C^1-foliations, secondary classes for C^2-foliations.

Linear algebra

$A \in GL(\mathbb{R}^q)$ acts on $\mathbb{R}^q$ via $L_A(\vec{v}) = A \cdot \vec{v}$.

There is well-known trichotomy for dynamics of linear actions:

• Elliptic, or isometric $\Leftrightarrow$ all orbits of L_A action are bounded $\Leftrightarrow$ $\|A^\ell\|$ is bounded as $\ell \to \infty$.

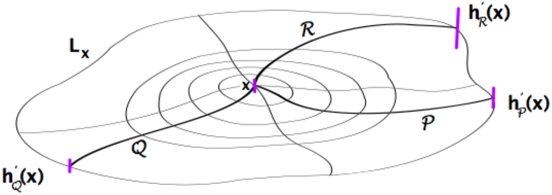
• Parabolic, or shear $\Leftrightarrow$ orbits of L_A action not bounded, but do not grow exponentially fast in norm $\Leftrightarrow$ $\|A^\ell\|$ grows at subexponential rate as $\ell \to \infty$.

• Hyperbolic, or partially hyperbolic $\Leftrightarrow$ some orbits of L_A action grow exponentially fast in norm $\Leftrightarrow$ $\|A^\ell\|$ grows at exponential rate as $\ell \to \infty$.

Asymptotic exponent

Definition: The *transverse expansion rate* function at x is

$$\lambda(\mathcal{G}_{\mathcal{F}}, k, x) = \max_{\|[g]\|_x \leq k} \frac{\ln\left(\max\{\|D_x g\|, \|(D_x g)^{-1}\|\}\right)}{k} \geq 0$$



Definition: The *asymptotic transverse growth rate* at x is

$$\lambda(\mathcal{G}_{\mathcal{F}}, x) = \limsup_{k \to \infty} \lambda(\mathcal{G}_{\mathcal{F}}, k, x) \geq 0$$

This is essentially the "maximum Lyapunov exponent" for $\mathcal{G}_{\mathcal{F}}$ at x.

$\lambda(\mathcal{G}_{\mathcal{F}}, x)$ is a Borel function of $x \in \mathcal{T}$, as each norm function $\|D_{w'} h_{\sigma_{w,z}}\|$ is continuous for $w' \in D(h_{\sigma_{w,z}})$ and the maximum of Borel functions is Borel.

Lemma: $\lambda_{\mathcal{F}}(z)$ is constant along leaves of $\mathcal{F}$.

$\Longrightarrow \mathcal{F}$ decomposes M into saturated Borel sets.

Expansion classification

Theorem: (Hurder, 2000, 2005) Let $\mathcal{F}$ be a C^1-foliation of compact manifold M. Then there is a disjoint decomposition

$$M = \mathcal{E}_{\mathcal{F}} \cup \mathcal{P}_{\mathcal{F}} \cup \mathcal{H}_{\mathcal{F}}$$

consisting of $\mathcal{F}$-saturated, Borel subsets of M, defined by:

1. Elliptic points: $\mathcal{E}_{\mathcal{F}} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \forall\ k \geq 0,\ \lambda(\mathcal{G}_{\mathcal{F}}, k, x) \leq \kappa(x)\}$
   i.e., "points of bounded expansion" (example: Riemannian foliations)
2. Parabolic points: $\mathcal{P}_{\mathcal{F}} \cap \mathcal{T} = \{x \in \mathcal{T} - (\mathcal{E}_{\mathcal{F}} \cap \mathcal{T}) \mid \lambda(\mathcal{G}_{\mathcal{F}}, x) = 0\}$
   i.e., "points of slow-growth expansion" (example: Distal foliations)
3. Hyperbolic points: $\mathcal{H}_{\mathcal{F}} \cap \mathcal{T} = \{x \in \mathcal{T} \mid \lambda(\mathcal{G}_{\mathcal{F}}, x) > 0\}$
   i.e., "points of exponential-growth expansion" (example: Anosov foliations, or more generally, non-uniformly, partially hyperbolic foliations)

Classification via transverse growth

Proposition: $\mathcal{F}$ a foliation with all leaves compact, then $\lambda(\mathcal{G}_{\mathcal{F}}, x) = 0$, $\forall\ x \in \mathcal{T}$.

The derivative cocycle is not trivial, but has "slow growth" as must be parabolic (upper triangular).

Theorem: If there exists $\kappa > 0$ so that $\lambda(\mathcal{G}_{\mathcal{F}}, k, x) \leq \kappa \quad \forall\ x \in \mathcal{T}$, then $\mathcal{F}$ is equicontinuous (and so Riemannian?)

The case where there is a bound $\kappa(x)$ depending on x, but there is no uniform upper bound, this remains a mystery!

The parabolic case of $\lambda(\mathcal{G}_{\mathcal{F}}, x) = 0$ for all $x \in \mathcal{T}$, but no pointwise upper bound, should be some sort of nil-geometry.

The case when the set $\mathcal{H}_{\mathcal{F}}$ of partially hyperbolic points is most interesting.

Godbillon-Vey classes

Assume that $\mathcal{F}$ is a C^2-foliation and the normal bundle $Q = TM/T\mathcal{F}$ is oriented.

Let ω be a q-form defining $T\mathcal{F}$, so then $d\omega = \eta \wedge \omega$ for a 1-form η.

The form $\eta \wedge (d\eta)^q$ is then a closed $2q+1$-form on M, and its cohomology class is independent of choices.

$$GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H^{2q+1}_{deR}(M)$$

is the *Godbillon-Vey class* of $\mathcal{F}$.

Question: (Moussu-Pelletier 1974, Sullivan 1975) If $GV(\mathcal{F}) \neq 0$, what does this imply about the dynamical properties of $\mathcal{F}$? Must there be leaves with exponential growth?

This question motivated many developments in the study of foliations.

Godbillon-Vey and exponent

Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^2-foliation of codimension-q with $GV(\mathcal{F}) \neq 0$. Then the hyperbolic set $\mathcal{H}_{\mathcal{F}}$ has *positive Lebesgue measure*.

Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^1-foliation of codimension-1 such that the hyperbolic set $\mathcal{H}_{\mathcal{F}}$ has positive Lebesgue measure. Then $\mathcal{F}$ has a resilient leaf, and hence has positive geometric entropy.

Corollary: (Duminy 1982) Let $\mathcal{F}$ be a C^2-foliation of codimension-1 with $GV(\mathcal{F}) \neq 0$. Then $\mathcal{F}$ has a resilient leaf.

Combining results, we have the stronger statement:

Theorem: (Hurder-Langevin 2004) Let $\mathcal{F}$ be a C^1-foliation of codimension-1 such that the hyperbolic set $\mathcal{H}_{\mathcal{F}}$ has positive Lebesgue measure. Then $\mathcal{F}$ has positive geometric entropy, in the sense of Ghys-Langevin-Walczak, so that $GV(\mathcal{F}) \neq 0$ implies positive geometric entropy.

Positive entropy and hyperbolicity

Questions for codimension $q > 1$:

Question: Suppose $\mathcal{G}_{\mathcal{F}}$ has positive geometric entropy. What can be said about the dynamics of $\mathcal{F}$, beyond the easy conclusion that $\lambda(\mathcal{G}_{\mathcal{F}}, x) > 0$ for some x?

Question: Suppose $\mathcal{H}_{\mathcal{F}}$ has positive Lebesgue. What can be said about the dynamics of $\mathcal{F}$? Must $\mathcal{G}_{\mathcal{F}}$ have positive entropy?

Question': Suppose $\mathcal{H}_{\mathcal{F}}$ has positive Lebesgue, and the normal derivative cocycle has *algebraic hull* $G \subset GL(\mathbb{R}^q)$ where G acts transitively on $\mathbb{R}^q$. Must $\mathcal{G}_{\mathcal{F}}$ have positive entropy?

These are all part of the general program to understand the structure of partially hyperbolic foliations.

Closed invariant sets

As an alternative to asking for the "big picture" of foliation dynamics, study the dynamics near closed invariant sets. This follows Smale's plan for the study of the dynamics of flows, where attractors are a central concept.

$\mathcal{Z} \subset M$ *minimal* $\Longleftrightarrow$ closed and every leaf $L \subset \mathcal{Z}$ is dense.

$\mathcal{W} \subset M$ is *transitive* $\Longleftrightarrow$ closed and there exists a dense leaf $L \subset \mathcal{W}$

M compact, then minimal sets for foliations always exist.

Transitive sets are most important for flows – Axiom A attractors are transitive sets, while the minimal sets include the periodic orbits in the domain of attraction.

Problem: Describe the minimal sets for $\mathcal{F}$ in each type of dynamic.

Restrict attention to *exceptional minimal sets* $\mathcal{Z}$, where $\mathcal{Z} \cap \mathcal{T}$ is a Cantor set.

$\mathfrak{M} \subset M$ has the structure of a *minimal matchbox manifold*; that is, it is a foliated space which is *transversally Cantor-like*.

Structure of exceptional minimal sets

Conjecture: If $\mathcal{F}$ is a C^2-foliation of codimension-one, and $\mathfrak{M} \subset M$ is an exceptional minimal set, then $\mathfrak{M}$ has the structure of a *Markov Minimal Set*.

That is, there is a covering of $\mathfrak{X} = \mathfrak{M} \cap \mathcal{T}$ by clopen sets, such that the restricted holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}^{\mathfrak{X}}$ is generated by a free semigroup.

Theorem: [Clark–H–Lukina, 2012] Let $\mathfrak{M}$ be an exceptional minimal set of dimension p for a C^r-foliation, $r \geq 0$. Then there exists a presentation $\mathcal{P} = \{p_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq \ell_0\}$ consisting of branched p-manifolds M_ℓ and local covering C^{r-1}-maps p_ℓ, such that $\mathfrak{M}$ is homeomorphic to the inverse system,

$$\mathfrak{M} \;\cong\; \mathcal{S}_{\mathcal{P}} = \varprojlim \; \{p_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq \ell_0\}.$$

This generalizes a well-known result for the case where $\mathfrak{M}$ is the tiling space associated to a repetitive, aperiodic tiling of $\mathbb{R}^q$ with finite local complexity.

Williams solenoids

Williams (1970, 1974) introduced a class of inverse limit spaces, called generalized solenoids, which model Smale's Axiom A attractors.

Let K be a branched manifold, i.e. each $x \in K$ has a neighborhood homeomorphic to the disjoint union of a finite number of Euclidean disks modulo some identifications.

$f : K \to K$ is an expansive immersion of branched manifolds satisfying a flattening condition. Then

$$\mathfrak{M} = \varprojlim \{f : K \to K\}$$

each point x has a neighborhood homeomorphic to $[-1,1]^n \times \mathrm{Cantor\ set}$. Moreover, f extends to a hyperbolic map defined on some open neighborhood of $\mathfrak{M} \subset V \subset \mathbb{R}^m$ for m sufficiently large. Thus, $\mathfrak{M}$ is an exceptional minimal set for the expanding foliation $\mathcal{F}_u$ of V for f, and $\mathfrak{M} \subset \mathcal{H}_{\mathcal{F}_u}$.

Parabolic minimal sets

Definition: A minimal set $\mathcal{Z}$ is said to be *parabolic* if $\mathcal{Z} \cap \mathcal{H}_{\mathcal{F}} = \emptyset$.

Proposition: Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.

Proof: If some holonomy transformation along L_w has a non-unitary eigenvalue, then it has a stable manifold.

Theorem: A parabolic minimal set has zero geometric entropy.

Question: What are the *zero entropy* exceptional minimal sets for C^r-foliations?

Solenoids and weak solenoids

Weak solenoids are generalizations to higher dimensions of 1-dimensional p-adic (Vietoris) solenoids, i.e. inverse limit of finite-to-one coverings of a circle

$$\mathbb{S}_\infty = \varprojlim \left\{ p_i : \mathbb{S}^1 \to \mathbb{S}^1,\ i \geq 0 \right\}.$$

An *n-dimensional solenoid*, as studied by McCord (1965) and Fokkink and Oversteegen (2002) is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \ \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps.

$\mathcal{S}$ is a fibre bundle with Cantor set fibre, and profinite structure group.

If all compositions $L_{\ell+1} \to L_0$ are normal (Galois) coverings, then $\mathcal{S}$ is called a McCord solenoid, and otherwise is a (weak) solenoid.

Embeddings

For these abstract solenoidal spaces, the basic question is whether they are homeomorphic to minimal sets of parabolic or hyperbolic type?

Embedding Property: Given a (generalized) solenoid $\mathcal{S}$, does there exists a C^r-foliation $\mathcal{F}_M$ of a compact manifold M and an embedding of $\iota \colon \mathcal{S} \hookrightarrow M$ as a foliated subspace? $(r \geq 0)$

Germinal Extension Property: Given a (generalized) solenoid $\mathcal{S}$, does there exists a C^r-foliation $\mathcal{F}_U$ of an open manifold U and an embedding of $\iota \colon \mathcal{S} \hookrightarrow U$ as a foliated subspace? $(r \geq 0)$

Solutions to the embedding problem for solenoids modeled on $\mathbb{S}^1$ were given by Gambaudo, Tressier, et al in 1990's.

Embeddings of toral solenoids

Theorem: [Clark–H, 2008] Let $\mathcal{F}_0$ be a C^r-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}$ on M which is C^r-close to $\mathcal{F}_0$, and $\mathcal{F}$ has a solenoidal minimal set contained in U with base L_0. If $\mathcal{F}_0$ is a distal foliation, then $\mathcal{F}$ is also distal.

The criteria for embedding depends on the degree of smoothness required, and the tower of subgroups of the fundamental group.

The study of minimal foliated spaces in the abstract, leads to very interesting (and seemingly difficult) problems concerning when these can be realized as minimal sets for C^r foliations, especially for $r \geq 2$.

Instability

One application is a type of "Reeb Instability" result:

Theorem: [Clark & H 2010] Let $\mathcal{F}_0$ be a C^∞-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some saturated open neighborhood U of L_0.

Then there exists a foliation $\mathcal{F}_M$ on M which is C^∞-close to $\mathcal{F}_0$, and $\mathcal{F}_M$ has an uncountable set of solenoidal minimal sets $\{\mathcal{S}_\alpha \mid \alpha \in \mathcal{A}\}$, which are *pairwise non-homeomorphic*.

Solenoid-type objects are "typical" for perturbations of dynamical systems formed by flows, and possibly also for foliations.

Expansive minimal sets

Theorem: [Auslander–Glasner–Weiss, 2007] Let Γ be a finitely generated group acting minimally on a Cantor set $\mathfrak{X}$. If the action is distal, then it must be equicontinuous.

This result suggests:

Conjecture: Let $\mathfrak{M} \subset M$ be an exceptional minimal set for a foliation $\mathcal{F}$. Then either the holonomy action of $\mathcal{G}_{\mathcal{F}}^{\mathfrak{X}}$ on $\mathfrak{X}$ equicontinuous, or it is expansive.

This brings the study of foliation dynamics full circle:

Problem: Let $\mathfrak{M} \subset M$ be an exceptional minimal set for a C^r-foliation $\mathcal{F}$ with $r \geq 1$, for which the holonomy action of $\mathcal{G}_{\mathcal{F}}^{\mathfrak{X}}$ on $\mathfrak{X}$ is expansive. Show that $\mathfrak{M} \cap \mathcal{H}_{\mathcal{F}} \neq \emptyset$, and give some sort of structure theory for these examples.

There exist examples of exceptional minimal sets for C^2-foliations with codimension-one, which contain parabolic leaves, so one cannot expect $\mathfrak{M} \subset \mathcal{H}_{\mathcal{F}}$.

The conclusion is that classifying foliations remains as elusive a goal as ever.

Though, the study of their dynamical properties provides the best method to date, for the passage from local definitions to global properties.

Thank you for your attention!