Foliation dynamics, shape and classification

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Theorem: [Denjoy, 1932] There exist a C^1-foliation $\mathcal{F}$ of codimension-1 with an exceptional minimal set $\mathfrak{M}$ with no fixed-points for its holonomy pseudogroup, and $\mathfrak{M}$ has the *shape* of wedge of two circles.

Theorem: [Sacksteder, 1965] An exceptional minimal set $\mathfrak{M}$ for a C^2-foliation of codimension-1 always has hyperbolic fixed-points for its holonomy pseudogroup.

Theorem: [Rosenberg & Roussarie, 1970] There exists C^∞-foliations of codimension-1 with exceptional minimal sets.

Theorem: [Williams, 1974] Let $\Lambda \subset M$ be an expanding attractor for a diffeomorphism $f \colon M \to M$, so that the *stable foliation* of f is C^1 on some open neighborhood of Λ. Then $f \colon \Lambda \to \Lambda$ is C^0-conjugate to the shift map on a *stationary* generalized solenoid.

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$\mathfrak{M} \subset M$ is exceptional minimal set - transversally is a Cantor set.

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Problem: classify the possible

- topological types of $\mathfrak{M}$, up to homeomorphism;
- holonomy pseudogroups for $\mathfrak{M}$, up to Morita equivalence;
- shapes of $\mathfrak{M}$, up to shape equivalence.

We consider this in a more general topological setting, and ask to what extent the Theorems above admit generalizations.

Definition: A *matchbox manifold* is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that the transverse model space $\mathfrak{X}$ is *totally disconnected*, and for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{X}_x \subset \mathfrak{X}$ is a clopen subset.

Figure: Blue tips are points in Cantor set $\mathfrak{X}_x$

Definition: $\mathfrak{M}$ is an *n-dimensional matchbox manifold* if:

• $\mathfrak{M}$ is a continuum, a compact, connected metric space;

• $\mathfrak{M}$ admits a covering by foliated coordinate charts
$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid 1 \leq i \leq \kappa\}$;

• each $\mathfrak{T}_i$ is a *clopen* subset of a *totally disconnected* space $\mathfrak{X}$.

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Transversal $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\kappa \subset \mathfrak{M}$ defined by coordinate charts.

Holonomy of $\mathcal{F}$ on $\mathcal{T} \Longrightarrow$ finitely generated pseudogroup $\mathcal{G}_\mathcal{F}$ acting by <u>homeomorphisms</u> defined on clopen subsets of $\mathfrak{X}$.

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, where the pseudogroup $\mathcal{G}_\mathcal{F}$ is the "fundamental group".

Metric properties:

Two metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are *Lipschitz equivalent*, if they satisfy a Lipschitz condition for some $C \geq 1$,

$$C^{-1} \cdot d_{\mathfrak{X}}(x,y) \ \leq \ d'_{\mathfrak{X}}(x,y) \ \leq \ C \cdot d_{\mathfrak{X}}(x,y) \quad \text{for all } x,y \in \mathfrak{X} \quad (1)$$

Proposition: Let $\mathcal{G}_{\mathfrak{X}}$ be a pseudogroup defined by the restriction of the holonomy for a minimal set $\mathcal{M}$ of a C^1-foliation to a transversal $\mathfrak{X} = \mathcal{M} \cap \mathcal{T}$, then $\mathfrak{X}$ has a metric $d_{\mathfrak{X}}$ for which the action of $\mathcal{G}_{\mathfrak{X}}$ is Lipschitz.

For a Cantor set defined as the space of ends of a tree with bounded complexity, there is a natural ultrametric.

For a “fractal” $\mathfrak{C}$ defined by an *Iterated Function System*, the tree associated to the pseudogroup action defines an ultrametric on $\mathfrak{C}$.

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Theorem: [Hurder, 2013] There exist compactly-generated pseudogroups $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, such that there is no metric on $\mathfrak{X}$ for which the generators of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipschitz condition.

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Construct pseudogroup adding elements with "ultra-contraction".

This construction yields pseudogroup analogs of the construction of open Riemannian manifolds which are not quasi-isometric to leaves of C^1-foliations by Attie & Hurder.

Definition: Let $\mathcal{G}_{\mathfrak{X}}$ and $\mathcal{G}_{\mathfrak{Y}}$ be a *minimal* pseudogroup actions on metric Cantor sets $(\mathfrak{X}, d_{\mathfrak{X}})$ and $(\mathfrak{Y}, d_{\mathfrak{Y}})$, respectively.

$(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X})$ and $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y})$ are *Morita equivalent* if there exist clopen subsets $V \subset \mathfrak{X}$ and $W \subset \mathfrak{Y}$, and a homeomorphism $h \colon V \to W$ which conjugates $\mathcal{G}_{\mathfrak{X}}|V$ to $\mathcal{G}_{\mathfrak{Y}}|W$.

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Problem: Classify the minimal Lipschitz pseudogroup actions on a Cantor set $\mathfrak{X}$, up to Lipschitz equivalence.

Problem: Given a compactly-generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, and suppose there exists a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators are Lipschitz, when is there a Lipschitz equivalence to the pseudogroup defined by an exceptional minimal set of a C^1 foliation?

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Lipschitz invariants of minimal Lipschitz pseudogroup actions:

- Hausdorff dimension
- Positive geometric entropy

Theorem: [Lukina, 2014] The Ghys-Kenyon construction of a metric Cantor set from the space of pointed subtrees of the Cayley graph of the free group $\mathbb{F}_n$ has *infinite* Hausdorff dimension, and admits a minimal Lipschitz pseudogroup action induced by $\mathbb{F}_n$.

Two dynamical properties of pseudogroup actions:

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $g \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(g) \text{ with } d_{\mathcal{T}}(x, x') < \delta \implies d_{\mathcal{T}}(g(x), g(c')) < \epsilon$$

Definition: $\mathfrak{M}$ is an ϵ*-expansive matchbox manifold* if it admits some covering by foliation charts, and there exists $\epsilon > 0$, so that for all $w \neq w' \in \mathcal{T}$ with $d_{\mathcal{T}}(w, w') < \epsilon$, there exists $g \in \mathcal{G}_{\mathcal{F}}$ defined by the holonomy along some path, with $w, w' \in \mathrm{Dom}(g)$ such that $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$.

Examples:

- The classical 1-dimensional solenoids are equicontinuous.
- Denjoy minimal sets are expansive.
- The tiling space for an aperiodic tiling of $\mathbb{R}^n$ with finite local complexity has expansive pseudogroup dynamics.

Need property of matchbox manifolds and their pseudogroup actions, which applies for both the usual examples, as well as the exceptional examples, which are not so exceptional.

A *continuum* is a compact connected metric space.

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Definition': The shape of a continuum Ω is defined by a collection of maps $\{q_\ell \colon \Omega \to Y_\ell \mid \ell = 1, 2, \ldots\}$ where each Y_ℓ is a finite simplicial complex, and each q_ℓ is a continuous onto map whose fibers have diameter bounded by ϵ_ℓ where $\epsilon_\ell \to 0$.

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Definition": [Alexandroff, 1928] Let $\mathcal{Y}$ be a collection of finite simplicial complexes. Then the shape of Ω is $\mathcal{Y}$-like, if for all $\epsilon > 0$, there exists $Y_\epsilon \in \mathcal{Y}$ and a map $q_\epsilon \colon \Omega \to Y_\epsilon$ such that the fibers of q_ϵ have diameters uniformly less than ϵ.

Definition: A continuum Ω is Y-like if it is $\mathcal{Y}$-like for $\mathcal{Y} = \{Y\}$. We say that Ω has the shape of Y.

Example: A classical 1-dimensional solenoid has the shape of $\mathbb{S}^1$.

Example: A Denjoy minimal set has the shape of $\mathbb{S}^1 \vee \mathbb{S}^1$.

Theorem: [Anderson & Putnam, 1999] The tiling space for an aperiodic tiling of $\mathbb{R}^n$ defined by a substitution rule has the shape of a branched n-manifold Y.

Presentation is a collection $\mathcal{P} = \{p_{\ell+1} \colon Y_{\ell+1} \to Y_\ell \mid \ell \geq 0\}$,

• each Y_ℓ is a connected compact simplicial complex, dimension n,

• each "bonding map" $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers.

Associated to $\mathcal{P}$ is the *generalized solenoid*

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon Y_{\ell+1} \to Y_\ell\} \; \subset \prod_{\ell \geq 0} Y_\ell$$

where $\mathcal{S}_{\mathcal{P}}$ is given the product topology.

$\mathcal{P}$ is *stationary* if $Y_\ell = Y_0$ for all $\ell \geq 0$, and the bonding maps $p_\ell = p_1$ for all $\ell \geq 1$.

Theorem: [Mardešić and Segal, 1963] Let Ω be a continuum which is $\mathcal{Y}$-like, then there exists $Y_\ell \in \mathcal{Y}$ and continuous surjections $p_{\ell+1}\colon Y_{\ell+1} \to Y_\ell$ for $\ell \geq 1$, such that Ω is homeomorphic to a generalized solenoid with these maps defining the presentation.

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Definition: $\mathcal{S}_\mathcal{P}$ is a *weak solenoid* if for each $\ell \geq 0$, M_ℓ is a compact manifold without boundary, and the map $p_{\ell+1}$ is a proper covering map of degree $m_{\ell+1} > 1$.

Theorem: [McCord, 1965] A weak solenoid is a matchbox manifold.

Remark: A generalized solenoid may be a matchbox manifold, such as for Williams solenoids which are stationary, and the Anderson–Putnam and Gähler–Sadun construction of finite approximations to tiling spaces.

Let $\mathfrak{M}$ be a minimal matchbox manifold $\mathfrak{M}$.

Consider the properties of $\mathfrak{M}$:

- Morita class of pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on Cantor set $\mathfrak{X}$
- Dynamical properties of $\mathcal{G}_{\mathcal{F}}$ acting on Cantor set $\mathfrak{X}$
- Shape of $\mathfrak{M}$
- $\mathfrak{M}$ is realized as exceptional minimal set of C^1-foliation
- Homeomorphism type of $\mathfrak{M}$

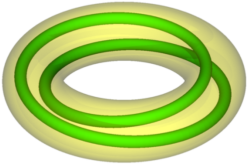
Program: Combine properties to obtain conclusions about others.

Theorem: [Hagopian, 1977] A matchbox manifold $\mathfrak{M}$ is $\mathbb{S}^1$-like if and only if it is homeomorphic to the inverse limit $\mathcal{S}_{\mathcal{P}}$ of a tower of coverings:

$$\longrightarrow \mathbb{S}^1 \xrightarrow{n_{\ell+1}} \mathbb{S}^1 \xrightarrow{n_\ell} \cdots \xrightarrow{n_2} \mathbb{S}^1 \xrightarrow{n_1} \mathbb{S}^1$$

where all covering degrees $n_\ell > 1$. These spaces were introduced by Vietoris [1927] and van Dantzig [1930].

If all $n_\ell = p > 1$ are constant, then $\mathcal{S}_{\mathcal{P}}$ can be realized as a Smale-Williams hyperbolic attractor for a smooth map of $\mathbb{S}^3$.



A continuum Ω is *homogeneous* if the group of homeomorphisms of Ω is *point-transitive*.

Theorem: [Clark & Hurder, 2010] Let $\mathfrak{M}$ be a matchbox manifold.

• If $\mathfrak{M}$ has equicontinuous pseudogroup, then $\mathfrak{M}$ is homeomorphic to a weak solenoid as foliated spaces.

• If $\mathfrak{M}$ is homogeneous, then $\mathfrak{M}$ has equicontinuous pseudogroup, and is moreover, homeomorphic to a McCord (normal) solenoid.

This last result is a higher-dimensional version of the *Bing Conjecture* for 1-dimensional matchbox manifolds.

The proof uses coding functions for the pseudogroup action on a transversal. Inspired by work of Coornaert & Papadopoulos in **Symbolic dynamics and hyperbolic groups**, 1993.

Let $\mathcal{M}_n = \{M \mid M \text{ is closed oriented manifold of dimension } n\}$

Theorem: [Clark, Hurder, Lukina, 2014] Let $\mathfrak{M}$ be a matchbox manifold which is $\mathcal{M}_n$-like. Then $\mathcal{G}_{\mathcal{F}}$ has equicontinuous dynamics, hence is homeomorphic to a weak solenoid.

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Theorem: [Clark, Hurder, Lukina, 2014] Let $\mathfrak{M}$ be a matchbox manifold which is $\mathcal{M}_n$-like. Then $\mathcal{G}_{\mathcal{F}}$ has equicontinuous dynamics, hence is homeomorphic to a weak solenoid.

Theorem: [Clark, Hurder, Lukina, 2013] Let $\mathfrak{M}$ be a matchbox manifold which is $\mathbb{T}^n$-like. Then $\mathfrak{M}$ is homeomorphic to an inverse limit space defined by a presentation

$$\longrightarrow \mathbb{T}^n \xrightarrow{p_{\ell+1}} \mathbb{T}^n \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} \mathbb{T}^n \xrightarrow{p_1} \mathbb{T}^n$$

Corollary: Let $\mathfrak{M}$ be a $\mathbb{T}^n$-like minimal matchbox manifold, then $\mathfrak{M}$ is homeomorphic to a minimal set of a C^1-foliation.

Recall the algebraic classification of $\mathbb{S}^1$-like solenoids.

Two sequences of primes $\mathcal{P} = (p_1, p_2, \ldots)$ and $\mathcal{Q} = (q_1, q_2, \ldots)$ are equivalent, $\mathcal{P} \sim \mathcal{Q}$, if and only if one can delete a finite number of entries from each sequence so that every prime occurs the same number of times in deleted sequences.

Example: $\mathcal{P} = (2, 3, 3, \ldots)$, $\mathcal{Q} = (3, 3, 3, \ldots)$, $\mathcal{S} = (2, 3, 2, 3, 2, 3, \ldots)$. Then $\mathcal{P} \sim \mathcal{Q}$, and $\mathcal{P} \nsim \mathcal{S}$, and $\mathcal{Q} \nsim \mathcal{S}$.

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Theorem: [Bing, 1960] $\mathcal{P} \sim \mathcal{Q} \Rightarrow \mathcal{S}_{\mathcal{P}}$ is homeomorphic to $\mathcal{S}_{\mathcal{Q}}$.

Theorem: [McCord, 1965] $\mathcal{P} \sim \mathcal{Q} \Leftarrow \mathcal{S}_{\mathcal{P}}$ is homeomorphic to $\mathcal{S}_{\mathcal{Q}}$.

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Remark: Results of Hjorth & Thomas imply there is no *algebraic* classification for weak solenoids, when dimension $n > 1$.

A manifold Y is *aspherical*, if it is connected and higher homotopy groups $\pi_n(X)$, $n \geq 2$ are trivial.

Equivalently, Y is *aspherical* if it is connected and has a contractible universal cover.

Example: $\mathbb{T}^n$, $n \geq 1$, surfaces of genus $g \geq 1$ are aspherical.

A sphere $\mathbb{S}^n$, $n \geq 2$, is not aspherical.

A collection $\mathcal{A}_B$ of closed manifolds is called *Borel* if it satisfies the following.

1. Each $Y \in \mathcal{A}_B$ is aspherical,
2. Any closed manifold X which is homotopy equivalent to a $Y \in \mathcal{A}_B$ is homeomorphic to Y,

A closed manifold Y is *strongly Borel* if the set of its finite covers forms a Borel collection.

Example: $\mathbb{T}^n$ is strongly Borel, as is every closed nilmanifold.

$n = 3$, Thurston Geometrization Conjecture, proved by Perleman.

$n = 4$, Freedman's proof of Poincaré Conjecture.

$n \geq 5$, Surgery Theory plus Poincaré Conjecture.

Theorem: [Clark, Hurder, Lukina, 2013] Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $\mathbb{T}^n$-like matchbox manifolds. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if $\mathcal{G}_{\mathcal{F}_1}$ and $\mathcal{G}_{\mathcal{F}_2}$ are Morita equivalent.

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Theorem: [Clark, Hurder, Lukina, 2013] Suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are Y-like matchbox manifolds, where Y is strongly Borel. Assume that each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ has a leaf which is simply connected. Then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are homeomorphic if and only if $\mathcal{G}_{\mathcal{F}_1}$ and $\mathcal{G}_{\mathcal{F}_2}$ are Morita equivalent.

The proofs of these two results are in our most recent paper,

Classifying matchbox manifolds, arXiv:1311.0226

and uses techniques from all previous works.

Some Problems

- Develop effective invariants for (Lipschitz) equicontinuous minimal pseudogroup actions on (metric) Cantor sets.

This is work in progress with Jessica Dyer and Olga Lukina.

- Let Y be a closed branched n-manifold. Give a "combinatorial" structure theorem for Y-like matchbox manifolds.

The case where the Y-like matchbox manifold $\mathfrak{M}$ has leaves with non-trivial holonomy is not well-understood.

- Given a Y-like matchbox manifold $\mathfrak{M}$, when does there exists a (Lipschitz) embedding into a C^r-foliation, for some $r \geq 1$?

Extend the results of [Williams, 1974] and [Clark-Hurder, 2011].

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Thank you for your attention.