Foliation dynamics, shape and classification
III. Classification Problems

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Let $\mathfrak{M}$ be a minimal matchbox manifold.

- Find invariants of the homeomorphism type.

Recall that $h\colon \mathfrak{M}^1 \to \mathfrak{M}^2$ a homeomorphism implies that h is a foliated homeomorphism, so classification is actually about classifying the foliation on $\mathfrak{M}$.

We can also ask for the weaker classification type:

- Find invariants of the topological orbit-equivalence type for $\mathfrak{M}$, or for the action of the foliation pseudogroup $\mathcal{G}_{\mathfrak{X}}$.

We consider the first question in this lecture.

Recall the Vietoris solenoid, defined by tower of coverings:

$$\mathcal{P} \;\equiv\; \cdots \longrightarrow \mathbb{S}^1 \xrightarrow{p_{\ell+1}} \mathbb{S}^1 \xrightarrow{p_\ell} \cdots \xrightarrow{p_2} \mathbb{S}^1 \xrightarrow{p_1} \mathbb{S}^1$$

where each p_ℓ is a covering map of degree $n_\ell > 1$. $\mathcal{P}$ is called a presentation, and the solenoid is given as the inverse limit

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \; \{p_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1\} \;\subset \prod_{\ell \geq 0} \mathbb{S}^1$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations, and let P be the infinite set of prime factors of the integers in the set $n_{\mathcal{P}} = \{n_1, n_2, n_3, \ldots\}$, included with multiplicity, and Q the same of $m_{\mathcal{P}} = \{m_1, m_2, m_3, \ldots\}$.

Theorem: [Bing, 1960; McCord, 1965] The solenoids $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic if and only if there is a bijection between a cofinite subset of P with a cofinite subset of Q.

The proof is based on a general result for inverse limits. Assume that we are given two presentations,

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \,,\ \mathcal{Q} = \{q_{\ell+1} \colon N_{\ell+1} \to N_\ell \mid \ell \geq 0\}$$

where all spaces $\{M_\ell \mid \ell \geq 0\}$ and $\{N_\ell \mid \ell \geq 0\}$ are compact oriented manifolds, and all bonding maps are orientation-preserving coverings. These define weak solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$.

Let $\Pi_\ell^\mathcal{P} \colon \mathcal{S}_\mathcal{P} \to M_\ell$ denote the fibration map onto the factor M_ℓ for $\mathcal{S}_\mathcal{P}$, and $\Pi_\ell^\mathcal{Q} \colon \mathcal{S}_\mathcal{Q} \to N_\ell$ that for $\mathcal{S}_\mathcal{Q}$.

Choose basepoints $\overline{x} \in \mathcal{S}_\mathcal{P}$ and $\overline{y} \in \mathcal{S}_\mathcal{Q}$. Then define basepoints $x_\ell = \Pi_\ell^\mathcal{P}(\overline{x}) \in M_\ell$ and $y_\ell = \Pi_\ell^\mathcal{Q}(\overline{y}) \in N_\ell$ for $\ell \geq 0$.

Theorem: [folklore] Given presentations $\mathcal{P}$ and $\mathcal{Q}$, suppose there exists increasing sequences of integers $\{0 \leq m_1 < m_2 < \cdots\}$ and $\{0 \leq n_1 < n_2 < \cdots\}$ and homeomorphisms

$$f_\ell \colon M_{m_\ell} \to N_{n_{\ell-1}} \quad , \quad g_\ell \colon N_{n_\ell} \to M_{m_\ell}$$

for $\ell \geq 1$, such that

- $f_\ell(x_{m_\ell}) = y_{n_{\ell-1}}$ and $g_\ell(y_{n_\ell}) = x_{m_\ell}$
- $f_\ell \circ g_\ell = q^{n_\ell}_{n_{\ell-1}} \colon N_{n_\ell} \to N_{n_{\ell-1}}$
- $g_\ell \circ f_{\ell+1} = p^{m_\ell}_{m_{\ell+1}} \colon M_{m_{\ell+1}} \to M_{m_\ell}$.

Then there is a homeomorphism $\widehat{f} \colon \mathcal{S}_\mathcal{P} \to \mathcal{S}_\mathcal{Q}$ with $\widehat{f}(\overline{x}) = \overline{y}$.

If the above conditions are satisfied, then we say that the presentations $\mathcal{P}$ and $\mathcal{Q}$ are (pointed) *tower equivalent*.

In order to use this result to prove that two weak solenoids are homeomorphic, it is necessary to check an infinite number of conditions to show the two spaces are homeomorphic.

In the case where all $M_\ell = N_\ell = \mathbb{S}^1$, then the classification of Vietoris solenoids follows, as there is a unique oriented homeomorphism from $\mathbb{S}^1$ to $\mathbb{S}^1$ up to isotopy, and an arbitrary covering of the circle is homeomorphic to a circle, so the only invariants to check are the covering degrees in the tower.

For manifolds of dimensions 2 or greater, it is also not so obvious how to show that the hypotheses of the theorem are satisfied.

Consider the example of normal solenoids defined by a tower of coverings of the torus $\mathbb{T}^n$. A covering of $\mathbb{T}^n$ is again a covering, which is defined up to homeomorphism by the map on fundamental groups,

$$(p_{\ell+1})_{\#} \colon \mathbb{Z}^n = \pi_1(\mathbb{T}^n, x_{\ell+1}) \to \pi_1(\mathbb{T}^n, x_\ell) = \mathbb{Z}^n$$

Given a presentation $\mathcal{P}$, define $q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0$

$$\Gamma_\ell = image\{(q_\ell)_{\#} \colon \pi_1(M_\ell, x_\ell) \to \pi_1(M_0, x_0)\}$$

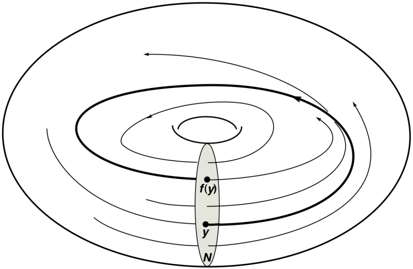
Thus, for a presentation $\mathcal{P}$ given by coverings of $\mathbb{T}^n$, the normal solenoid $\mathcal{S}_{\mathcal{P}}$ is characterized by a tower of descending subgroups of finite index, $\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$, where $\Gamma_\ell \cong \mathbb{Z}^n$.

For $n \geq 2$, the classification of such infinite descending chains is a topic of Descriptive Set Theory, and such chains they are far too complicated to be "classified". [Kechris, 2000], [Thomas, 2002]

There is an alternate characterization of the Vietoris solenoids, due to [Aarts and Fokkink, 1991].

Recall that a section C for a smooth non-singular flow, $\varphi\colon \mathbb{R} \times M \to M$, is an embedded submanifold, possibly with boundary, $\tau\colon N \to M$ such that:

- $dim(N) = dim(M) - 1$;
- the image of τ is everywhere transverse to the flow;
- the φ flow of every point $x \in M$ intersects the interior of $\tau(N)$.



For a matchbox manifold, the definition of a section C requires more care, as there is no "natural" notion of transversality.

Aarts and Fokkink use the fibration structure of the Vietoris solenoid, $\Pi\colon \mathcal{S}_{\mathcal{P}} \to \mathbb{S}^1$, where the fiber is identified with the product space

$$\Pi^{-1}(x_0) = \prod_{\ell=1}^{\infty} \{\mathbb{Z}/m_\ell \cdot \mathbb{Z}\}$$

The flow on the solenoid in minimal, so each cylinder set is a clopen section, for $\ell_0 \geq 1$,

$$C = (i_1, i_2, \ldots, i_{\ell_0 - 1}) \times \prod_{\ell=\ell_0}^{\infty} \{\mathbb{Z}/m_\ell \cdot \mathbb{Z}\}$$

Introduce the first return mapping $\varphi_C\colon C \to C$.

Given a second presentation $\mathcal{Q}$ with associated set of covering degrees $m_{\mathcal{Q}} = \{m_1, m_2, \dots\}$, let ψ denote the flow on $\mathcal{S}_{\mathcal{Q}}$, and for a cylinder set C' consider the induced flow $\psi_{C'} \colon C' \to C'$.

Definition: The flows φ on $\mathcal{S}_{\mathcal{P}}$ and ψ on $\mathcal{S}_{\mathcal{Q}}$ are return equivalent, if there exists sections C and C' as above, and a homeomorphism $h \colon C \to C'$, such that $h \circ \varphi_C = \psi_{C'} \circ h$.

Theorem: [Aarts and Fokkink, 1991] Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations for Vietoris solenoids, with indexing sets P and Q respectively, consisting of primes. Then $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic $\Leftrightarrow$ they are return equivalent $\Leftrightarrow$ there is a bijection between a cofinite subset of P with a cofinite subset of Q.

The proof that $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{Q}}$ are homeomorphic $\Leftrightarrow$ they are return equivalent, uses simply that each space is homeomorphic to the suspension of its the return map.

The proof that φ_C and $\psi_{C'}$ are conjugate, for some choice of sections by cylinder $\Leftrightarrow$ there is a bijection between a cofinite subset of P with a cofinite subset of Q, is a nice exercise.

Let $\mathfrak{M}$ be a matchbox manifold, and assume there is a choice of a regular covering by foliated coordinate charts,

$$\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq k\}$$

Identify $\mathfrak{T}_i = \varphi_i^{-1}(0 \times \mathfrak{X}_i) \subset \mathfrak{M}$ with $\mathfrak{X}_i$ and thus $\mathfrak{X}$ with $\mathfrak{T} \subset \mathfrak{M}$.

For $x \in \mathfrak{M}$ let L_x be the leaf of $\mathcal{F}$ containing x.

Definition: A (regular) section for $\mathfrak{M}$ is a clopen subset $C \subset \mathfrak{T}$ such that for each $x \in M$, the intersection $L_x \cap C \neq \emptyset$.

If $\mathfrak{M}$ is minimal, then any clopen subset $C \subset \mathfrak{T}$ is a section.

Let $\mathcal{G}_{\mathfrak{X}}$ be the pseudogroup for $\mathfrak{M}$ associated to the covering, acting on the Cantor set $\mathfrak{X}$. Given a regular section $C \subset \mathfrak{T}$, the induced pseudogroup is defined by

$$\mathcal{G}_C = \{h \in \mathcal{G}_{\mathfrak{X}} \mid \mathrm{Dom}(h) \subset C, \mathrm{Range}(h) \subset C\}$$

where we identify C with its image in $\mathfrak{X}$.

Definition: Matchbox manifolds $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are *return equivalent* if there exists

- regular coverings of $\mathfrak{M}^1$ and $\mathfrak{M}^2$ by foliated coordinate charts, with transversals $\mathfrak{T}^1$ and $\mathfrak{T}^2$,
- pseudogroups $\mathcal{G}_{\mathfrak{X}^1}$ and $\mathcal{G}_{\mathfrak{X}2}$, respectively,
- sections $C^1 \subset \mathfrak{T}^1$ and $C^2 \subset \mathfrak{T}^2$,
- a homeomorphism $h\colon C^1 \to C^2$ which conjugates $\mathcal{G}_{C^1}$ to $\mathcal{G}_{C^2}$.

That is, for all $g \in \mathcal{G}_{C^1}$ we have $h \circ g \circ h^{-1} \in \mathcal{G}_{C^2}$, and vice-versa.

Proposition: Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be a homeomorphic minimal matchbox manifolds, with regular sections C^1, C^2. Then the induced pseudogroups $\mathcal{G}_{C^1}$ to $\mathcal{G}_{C^2}$ are return equivalent.

Thus, the return equivalence is a homeomorphism invariant for the class of minimal matchbox manifolds.

Without the assumption that the foliations are minimal, there are examples which show that return equivalence is not transitive.

Conjecture: Minimal matchbox manifolds $\mathfrak{M}^1$ and $\mathfrak{M}^2$ with the same leaf dimension are homeomorphic if and only if they are return equivalent.

Sadly, this is very false, for several basic reasons.

First, consider a minimal action $\varphi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$, where Γ is a finitely generated group.

Let M_0, N_0 be closed n manifolds such that there exists surjections of their fundamental groups

$$\rho_1\colon \pi_1(M_0, x_0) \to \Gamma \quad , \quad \rho_2\colon \pi_1(N_0, y_0) \to \Gamma$$

The suspension construction gives minimal matchbox manifolds with conjugate holonomy pseudogroups

$$\begin{aligned}\mathfrak{M}^1 &= \widetilde{M}_0 \times \mathfrak{X}/(w,x) \sim (w\cdot\gamma, \rho_1(\gamma^{-1})\cdot x)\ ,\ \gamma \in \pi_1(M_0, x_0)\\ \mathfrak{M}^2 &= \widetilde{N}_0 \times \mathfrak{X}/(w,x) \sim (w\cdot\gamma, \rho_1(\gamma^{-1})\cdot x)\ ,\ \gamma \in \pi_1(N_0, y_0)\end{aligned}$$

where $\widetilde{M}_0$ is the universal overing of M_0, and $\pi_1(M_0, x_0)$ acts on the right on $\widetilde{M}$ by deck translation, and similarly $\widetilde{N}_0$ is the universal covering for N_0 with the right action of $\pi_1(N_0, y_0)$.

But unless some covering of M_0 is homeomorphic to some covering of N_0, these spaces can not be homeomorphic.

For example, let $M_0 = \mathbb{T}^2$ and $N_0 = \Sigma_2$ be the surface of genus 2, and use any minimal action φ of $\mathbb{Z}^2$ on a Cantor set $\mathfrak{X}$.

On the other hand, we have the following generalization of the result of Aarts and Fokkink:

Theorem: Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be weak solenoids, with presentations $\mathcal{P}$ and $\mathcal{Q}$ with base manifolds $M_0 = N_0 = \mathbb{T}^n$. Then $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are return equivalent if and only if they are homeomorphic.

The proof uses two key properties of the torus $\mathbb{T}^n$, which are trivially true for the test case with $M_0 = \mathbb{S}^1$.

Definition: A manifold Y is *aspherical* if its universal covering $\widetilde{Y}$ is a contractible space.

We also require that M_0 is *strongly Borel*:

Definition: A closed manifold Y is *strongly Borel* if the collection $\mathcal{A}_Y \equiv \langle Y \rangle$ of all finite covers of Y forms a Borel collection. That is, it satisfies the conditions:

- Each $Y \in \mathcal{A}_B$ is aspherical,
- Any closed manifold X homotopy equivalent to some $Y \in \mathcal{A}_B$ is homeomorphic to Y, and

Examples of strongly Borel closed manifolds include the torus $\mathbb{T}^n$ for all $n \geq 1$, all closed *infra-nilmanifolds*, and all closed Riemannian manifolds Y with negative sectional curvatures.

Theorem: Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be equicontinuous matchbox manifolds which are Y-like, where Y is a strongly Borel closed manifold. Assume that both $\mathfrak{M}^1$ and $\mathfrak{M}^2$ have a leaf which is simply connected. Then $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are return equivalent if and only if they are homeomorphic.

We sketch the proof of this result for the case $Y = \mathbb{T}^n$, and with the assumption that each space has a simply connected leaf, as this will illustrate the key ideas and issues.

We assume there are given presentations

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}\,,\ \mathcal{Q} = \{q_{\ell+1} \colon N_{\ell+1} \to N_\ell \mid \ell \geq 0\}$$

For the case where $M_0 = N_0 = \mathbb{T}^n$, then all the spaces considered are coverings of $\mathbb{T}^n$ hence are again $\mathbb{T}^n$.

Let $\Pi_\ell^{\mathcal{P}} \colon \mathcal{S}_{\mathcal{P}} \to M_\ell$ denote the fibration map onto the factor M_ℓ for $\mathcal{S}_{\mathcal{P}}$, and $\Pi_\ell^{\mathcal{Q}} \colon \mathcal{S}_{\mathcal{Q}} \to N_\ell$ that for $\mathcal{S}_{\mathcal{Q}}$.

Choose basepoints $\overline{x} \in \mathcal{S}_{\mathcal{P}}$ and $\overline{y} \in \mathcal{S}_{\mathcal{Q}}$ and define basepoints $x_\ell = \Pi_\ell^{\mathcal{P}}(\overline{x}) \in M_\ell$ and $y_\ell = \Pi_\ell^{\mathcal{Q}}(\overline{y}) \in N_\ell$ for $\ell \geq 0$.

Let $\Lambda_1 = \pi_1(M_0, x_0)$ and $\Lambda_2 = \pi_1(N_0, y_0)$.

Introduce the Cantor fibers:

- $\mathfrak{X}^1 = \Pi_1^{-1}(x_0) \subset \mathfrak{M}^1$ where $\Pi_1 \colon \mathfrak{M}^1 \to M_0$
- $\mathfrak{X}^2 = \Pi_2^{-1}(y_0) \subset \mathfrak{M}^2$ where $\Pi_2 \colon \mathfrak{M}^2 \to N_0$

Then the fibration structure for $\mathfrak{M}^i$ defines a holonomy representation $\rho_i \colon \Lambda_i \to \mathrm{Homeo}(\mathfrak{X}^1)$, and $\mathfrak{M}^i$ is homeomorphic to the suspension space for this action on $\mathfrak{X}^1$, for $i = 1, 2$.

The assumption that each space $\mathfrak{M}_i$ has a simply connected leaf implies that the map ρ_i is injective.

The assumption that $\mathfrak{M}^1$ is return equivalent to $\mathfrak{M}^2$ implies that there exists clopen subsets $C^i \subset \mathfrak{X}^i$ and a homeomorphism $h\colon C^1 \to C^2$ which conjugates the holonomy pseudogroups $\mathcal{G}_{C^1}$ and $\mathcal{G}_{C^2}$.

Lemma: For $i = 1, 2$, there exists choices of C^i such that

$$\mathcal{G}_{C^i} = \Lambda_{C^i} = \{g \in \Lambda_i \mid \rho_i(C^i) = C_i\}$$

Moreover, the restricted map $\rho_i \colon \Lambda_{C^i} \to \mathrm{Homeo}(C^i)$ is injective.

The proof of this uses the usual properties of clopen sets in a Cantor set.

This implies that the conjugating map h induces an isomorphism between the images $\rho_i(\Lambda_{C^i}) \subset \mathrm{Homeo}(C^i)$.

Thus, h induces an isomorphism $h_{\#} \colon \Lambda_{C^1} \to \Lambda_{C^2}$.

Recall that $\Lambda_1 = \pi_1(M_0, x_0)$ and $\Lambda_2 = \pi_1(N_0, y_0)$. Thus,

- $\Lambda_{C^1} \subset \Lambda_1$ defines a covering space $M_0' \to M_0$
- $\Lambda_{C^2} \subset \Lambda_2$ defines a covering space $N_0' \to N_0$

By the assumption that M_0 and N_0 are $\mathbb{T}^n$ (or better, that they are strongly Borel) the isomorphism $h_\#$ between the fundamental groups of their coverings induces a homeomorphism $h' \colon M_0' \to N_0'$ which is compatible with the actions of the fundamental groups on the fibers C_1 and C_2.

The conclusion that $\mathfrak{M}^1$ is homeomorphic to $\mathfrak{M}^2$ now follows:

- $\mathfrak{M}^i$ is homeomorphic to the suspension of the action ρ_i on the space C^i by the fundamental groups Λ_{C^i}.

- The homeomorphisms $h' \colon M_0' \to N_0'$ and $h \colon C^1 \to C^2$ define a foliated bundle isomorphism, hence a homeomorphism of their suspension spaces.

This above sketch of proof skips many technical details.

Sketch reveals the parallels with the proof in [Aarts and Fokkink, 1991]. See [Clark, Hurder, Lukina, 2013] for all gory details, and other interesting points in the proof.

For example, in the case where $Y = \mathbb{T}^n$ and the spaces are Y-like, the assumption that there is a simply connected leaf can be omitted, due to the simple algebraic structure of $\mathbb{Z}^n$.

It is possible this assumption can be omitted in all cases, as the hypothesis that the spaces are Y-like may imply the algebraic splitting condition that is required.

It is natural to ask if the result remains true if Y is a branched manifold. For example, [Kwapisz, 2011] proves such a conjugation result for the tiling spaces associated to substitution tilings of $\mathbb{R}^n$.

Definition: A matchbox manifold $\mathfrak{M}$ is an *expanding* if there exists a self-homeomorphism $h\colon \mathfrak{M} \to \mathfrak{M}$ which is strictly expanding on leaves.

Conjecture: Let $\mathfrak{M}^1$ and $\mathfrak{M}^2$ be Y-like and expanding, where Y is a branched manifold. Then $\mathfrak{M}^1$ and $\mathfrak{M}^2$ are homeomorphic if and only if they are return equivalent.

This seems likely to be true.

What is certainly true, is that the classification problem requires more invariants of return equivalence for Cantor pseudogroups.

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