Dynamical Commensurator Groups

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Let $\mathfrak{M}$ be a compact, connected metric space (a continuum).

$\mathrm{Homeo}(\mathfrak{M})$ denotes the homeomorphisms of $\mathfrak{M}$.

$\mathrm{Homeo}_0(\mathfrak{M})$ denotes the closed subgroup of $\mathrm{Homeo}(\mathfrak{M})$ which are isotopic to the identity map, a contractible space.

Definitions:

- [Mapping Class Group] $\mathrm{Mod}(\mathfrak{M}) \equiv \mathrm{Homeo}(\mathfrak{M})/\mathrm{Homeo}_0(\mathfrak{M})$
- [Pointed Mapping Class Group:] $\widehat{x} \in \mathfrak{M}$, set

$$\mathrm{Mod}(\mathfrak{M}, \widehat{x}) \equiv \mathrm{Homeo}(\mathfrak{M}, \widehat{\mathrm{x}})/\mathrm{Homeo}_0(\mathfrak{M}, \widehat{\mathrm{x}})$$

Problem: Calculate $\mathrm{Mod}(\mathfrak{M}, \widehat{x})$ for $\mathfrak{M}$ a *solenoidal manifold*.

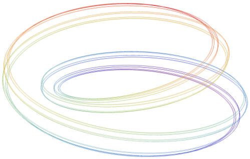
Strategy: Find algebraic model for $\mathrm{Mod}(\mathfrak{M}, \widehat{x})$

The most basic example - the Vietoris solenoid:

Choose a sequence of integers $\vec{m} = (m_1, m_2, \ldots)$ with $m_\ell > 1$.
Form the tower of coverings

$$\mathbb{S}^1 \xleftarrow{m_1} \mathbb{S}^1 \xleftarrow{m_2} \mathbb{S}^1 \xleftarrow{m_3} \mathbb{S}^1 \xleftarrow{m_4} \cdots$$

$$\mathfrak{M}_{\vec{m}} = \varprojlim \, \{m_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1\} \subset \prod_{\ell \geq 0} \mathbb{S}^1$$



- Let M be a compact connected manifold without boundary.
- Let $\Gamma = \pi_1(M, x)$ for choice of $x \in M$
- Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a descending chain of subgroups of finite index. The chain $\mathcal{G}$ and basepoint $x \in M$ determine a tower of finite index coverings by closed manifolds:

$$\mathcal{P}_{\mathcal{G}} \equiv \qquad \{M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} \cdots\}$$

where $q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \to M_0$ induces the map

$$(q_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to \Gamma_\ell \subset \pi_1(M_0, x) = \Gamma_0$$

$$\mathfrak{M}_{\mathcal{P}} = \varprojlim \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell$$

Definition: $\mathfrak{M}_{\mathcal{P}}$ is a *weak solenoid* (McCord, 1965) or a *solenoidal manifold* (Sullivan, 2014)

- A point $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ is a sequence $\widehat{x} = (x_0, x_1, x_2, \ldots)$ where $x_\ell \in M_\ell$ and $p_\ell(x_{\ell+1}) = x_\ell$ for $\ell \geq 0$.

- The map $\pi_k(x_0, x_1, x_2, \ldots) = (x_k, x_{k+1}, \ldots)$ is a homeomorphism.

- $\mathfrak{M}_{\mathcal{P}}$ is a compact connected foliated (laminated) space. The leaves are the connected components of $\mathfrak{M}_{\mathcal{P}}$

- $\widehat{q}_0 \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$ defined by $\widehat{q}_\ell(\widehat{x}) = x_\ell$ is a fibration.

- Each fiber $\mathfrak{X}_\ell = \widehat{q}_\ell^{-1}(x_\ell)$ a Cantor space.

Theorem: A pointed homeomorphism $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ is ϵ-homotopic to a map induced from a pointed zigzag map, for increasing sequence $k_i \geq i$,

$$\begin{array}{ccccccccc}
M_0 & \longleftarrow & M_{k_0} & \longleftarrow & M_{k_1} & \longleftarrow & M_{k_2} & \longleftarrow & \cdots \\
 & \swarrow{\scriptstyle \psi_{i_0}} & & \swarrow{\scriptstyle \psi_{i_1}} & & \swarrow{\scriptstyle \psi_{i_2}} & & & \\
M_0 & \longleftarrow & M_1 & \longleftarrow & M_2 & \longleftarrow & M_3 & \longleftarrow & \cdots
\end{array}$$

Corollary: Let $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ be a homeomorphism preserving a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$. Then h induces an injection $h_{\#}\colon \Gamma_{k_i} \to \Gamma_j$ whose image has finite index in Γ.

$\star$ Rogers & Tollefson, *Homeomorphisms homotopic to induced homeomorphisms of weak solenoidal spaces*, **Colloq. Math.** 25:81–87, 1972.

Let Γ be a countable group.

A *commensurator* of Γ is a pair of finite-index subgroups $H, K \subset \Gamma$ and an isomorphism $\phi\colon H \to K$.

Two commensurators $\phi_1\colon H_1 \to K_1$ and $\phi_2\colon H_2 \to K_2$ are equivalent, $\phi_1 \sim \phi_2$, if there exists a finite index subgroup $H_3 \subset H_1 \cap H_2$ such that $\phi_1|H_3 = \phi_2|H_3$.

Definition: The *abstract commensurator group* $\mathrm{Comm}(\Gamma)$ is the collection of commensurators $\phi\colon H \to K$, modulo $\sim$.

Remark: Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Given a commensurator $\phi\colon H \to K$ in $\mathrm{Comm}(\Gamma)$, observe that $\phi\colon H \cap \Gamma' \to K \cap \Gamma'$ is a commensurator in $\mathrm{Comm}(\Gamma')$. The converse clearly holds, hence $\mathrm{Comm}(\Gamma') \cong \mathrm{Comm}(\Gamma)$.

$\mathrm{Comm}(\Gamma)$ can be intuitively viewed as the group of "germs" of isomorphisms between finite-index subgroups of G.

For Γ finitely-generated, it may happen that $\mathrm{Comm}(\Gamma)$ is not finitely generated, as for $\Gamma = \mathbb{Z}$. Or it may happen that $\mathrm{Comm}(\Gamma) = \Gamma$, as for the Mapping Class Group $\Gamma = \mathrm{Mod}(\Sigma_g)$.

⋆ D. Studenmund, *Abstract commensurators: a survey of constructions and computations*, in "Geometric methods in group theory: papers dedicated to Ruth Charney" , 243–260, Sémin. Congr., 34, Soc. Math. France, Paris. 2024

Rogers & Tollefson Theorem yields:

Theorem: $\mathfrak{M}_{\mathcal{P}}$ a solenoid with base manifold M, $\Gamma = \pi_1(M, x)$. Then there is a natural map $\chi \colon \mathrm{Mod}(\mathfrak{M}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$.

Example: $\mathrm{Comm}(\mathbb{Z}) = \mathbf{GL}(\mathbb{Q}) = \mathbb{Q}^*$ the non-zero rationals

Let $\widehat{\mathbb{S}^1}$ denote the universal solenoid over $\mathbb{S}^1$. That is, the inverse limit of all finite coverings of $\mathbb{S}^1$.

Given $m/n \in \mathbb{Q}^*$ relatively prime, let $H = n \cdot \mathbb{Z} \subset \mathbb{Z}$ and $K = m \cdot \mathbb{Z}$. Then $\phi_{m/n} \colon H \to K$ is an isomorphism.

The map $\phi_{m/n}$ induces a map of the n-fold covering $\mathbb{S}^1 = \mathbb{R}/n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ to the m-fold covering $\mathbb{S}^1 = \mathbb{R}/m\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Thus, $\phi_{m/n}$ induces a homeomorphism of $\widehat{\mathbb{S}^1}$, and we obtain:

Theorem: $\chi \colon \mathrm{Mod}(\widehat{\mathbb{S}^1}, \widehat{x}) \cong \mathrm{Comm}(\mathbb{Z}) \cong \mathbb{Q}^*$

Example: Let Σ be a surface of genus $g \geq 2$, and $x \in \Sigma$.

Let $\widehat{\Gamma}$ denote the profinite completion of $\Gamma = \pi_1(\Sigma_g, x)$.

Let $\Gamma = \pi_1(\Sigma, x)$ then $\phi \in \mathrm{Comm}(\Gamma)$ is represented by an isomorphism $\phi\colon H \to K$ where $H, K \subset \Gamma$ have finite-index.

Let $\Sigma_H \to \Sigma$ be the covering defined by the subgroup H, and likewise for Σ_K. Then ϕ induces a pointed homeomorphism $h_\phi\colon \Sigma_H \to \Sigma_K$ with $h_\phi(x_H) = x_K$.

Let $\widehat{\Sigma_g}$ be the *universal hyperbolic solenoid*. $\widehat{\Sigma_g}$ is the solenoidal manifold defined by the collection of all finite coverings of Σ_g.

Theorem: $\chi\colon \mathrm{Mod}(\widehat{\Sigma_g}, \widehat{x}) \cong \mathrm{Comm}(\Gamma)$

$\star$ Odden, *The baseleaf preserving mapping class group of the universal hyperbolic solenoid*, **Trans. Amer. Math. Soc.**, 357:1829–1858, 2005.

Example: Let M be a closed aspherical manifold, $\widehat{x} \in M$

Let $\widehat{\Gamma}$ denote the profinite completion of $\Gamma = \pi_1(\Sigma_g, x)$.

Let $\widehat{M}$ denote the universal solenoid over M.

Let $\mathcal{E}(\widehat{M}, \widehat{x})$ be the pointed homotopy self-equivalences of $\widehat{M}$.

Theorem: $\chi \colon \mathcal{E}(\widehat{M}, \widehat{x}) \cong \mathrm{Comm}(\Gamma)$

⋆ Bering & Studenmund, *Topological models of abstract commensurators*, **Groups Geom. Dyn.**, 18:1403–1425, 2024.

Question: What about $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ for a general solenoid $\mathfrak{M}_{\mathcal{P}}$?

- The map χ need not be onto, as ϕ_h must preserve $\mathcal{G}_{\mathcal{P}}$
- The map χ need not be injective.

We introduce methods from dynamical systems to study χ.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain in Γ.

For $\ell \geq 0$ let $X_\ell = \Gamma/\Gamma_\ell$ as a left Γ-space.

- Γ acts transitively on finite set $X_\ell = \Gamma/\Gamma_\ell$

$$X_{\mathcal{G}} \equiv \varprojlim \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell$$

$$x = (x_0, x_1, \ldots) \in X_{\mathcal{G}} \iff p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0$$

Basic open sets are defined by fixing finite number of entries.

Γ acts on $X_{\mathcal{G}}$ by acting on each factor X_ℓ.

The action $\Phi \colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ is a (generalized) *odometer*. That is, it is a minimal equicontinuous action on a Cantor space.

A clopen set $U \subset X_{\mathcal{G}}$ is *adapted* if $U \neq \emptyset$, and for all $\gamma \in \Gamma$, if $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma \cdot U = U$.

$\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$ is a subgroup of finite index in Γ.

Definition. Odometers $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ and $\Phi' \colon \Gamma' \times \mathfrak{X}' \to \mathfrak{X}'$ are *return equivalent* if there exists adapted sets $U \subset \mathfrak{X}$ and $U' \subset \mathfrak{X}'$ and homeomorphism $h_U \colon U \to U'$ that conjugates the subgroups

$$\begin{aligned} \mathcal{H}_U &= \mathrm{Image}\{\Phi_U \colon \Gamma_U \to \mathrm{Homeo(U)}\} \\ \mathcal{H}'_{U'} &= \mathrm{Image}\{\Phi'_{U'} \colon \Gamma'_{U'} \to \mathrm{Homeo(U')}\} \end{aligned}$$

The map $\Phi_U \colon \Gamma_U \to \mathcal{H}_U$ may have kernel, and likewise for $\Phi'_{U'}$.

Thus, a homeomorphism $h_U \colon U \to U'$ which conjugates $\mathcal{H}_U$ with $\mathcal{H}'_{U'}$ need not induce an isomorphism $\phi_U \colon \Gamma_U \to \Gamma'_{U'}$.

Definition: Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer action, and $x \in \mathfrak{X}$.

- A *dynamical commensurator* is a homeomorphism $h_U \colon U \to V$, where U and V are adapted subsets with $x \in U \cap V$, such that $h_U(x) = x$, and h_U induces an isomorphism $\Theta_U \colon \mathcal{H}_U \to \mathcal{H}_V$.

- Commensurators $h_U \colon U \to V$ and $h_{U'} \colon U' \to V'$ are equivalent if there exists and adapted set U'' with $x \in U'' \subset U \cap U'$ such that $h_U | U'' = h'_{U'} | U''$. We write $(h_U, U, V) \overset{c}{\sim} (h'_{U'}, U', V')$.

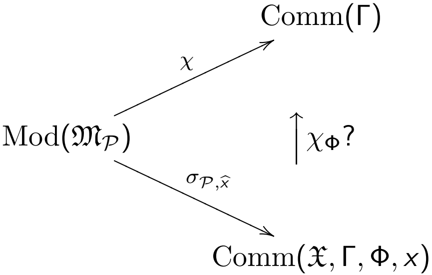
Definition: The *dynamical commensurator group* of $(\mathfrak{X}, \Gamma, \Phi)$ at x is the set of germs of dynamical commensurators,

$$\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) = \{h \colon U \to V \mid x \in U \cap V\} / \overset{c}{\sim}$$

Theorem. Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are weak solenoids. If $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic, then their monodromy odometers $\Phi \colon \Gamma \times X_{\mathcal{P}} \to X_{\mathcal{P}}$ and $\Phi' \colon \Gamma' \times X'_{\mathcal{P}'} \to X'_{\mathcal{P}'}$ are *return equivalent*.

⋆ Clark, Hurder & Lukina, *Classifying matchbox manifolds*, *Geom. Topol.*, 23(1):1–27, 2019.

Theorem: Let $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold, with monodromy action $(\mathfrak{X}, \Gamma, \Phi)$. For $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$, there there are well-defined maps



Problem: How is $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$ related to $\mathrm{Comm}(\Gamma)$?

The action $\Phi \colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ is *effective* if the action map $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has trivial kernel.

Consider the commensurator group relative to $\mathrm{Homeo}(\mathfrak{X})$:

$$\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma) = \{h \in \mathrm{Homeo}(\mathfrak{X}) \mid \mathrm{h} \colon \Phi(\Gamma) \cong \Phi(\mathrm{K}); \mathrm{H}, \mathrm{K} \subset_{\mathrm{f}} \Gamma\}$$

So we have

$$\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \ , \ \mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma) \ , \ \mathrm{Comm}(\Gamma)$$

For an effective action Φ, clearly $\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma) \subset \mathrm{Comm}(\Gamma)$.

The distinction between $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$ and $\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma)$ is in the domains of the conjugating maps.

Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an odometer. The action is:

- *quasi-analytic* if for each clopen set $U \subset \mathfrak{X}$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on $\mathfrak{X}$.

(The dynamical analog of the *Unique Root Property*.)

- *locally quasi-analytic* if there exists $\epsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$, and for any non-empty open subset $V \subset U$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on U.

(A localized *Unique Root Property*.)

- An odometer $(\mathfrak{X}, \Gamma, \Phi)$ is *coherent* if for every adapted set $U \subset \mathfrak{X}$, the restricted holonomy action map $\Phi_U \colon \Gamma_U \to \mathit{Homeo}(U)$ has finite kernel.

Example: If Γ is virtually nilpotent, that is, Γ contains a nilpotent subgroup $\Gamma' \subset \Gamma$ with finite index, then an effective odometer action of Γ is coherent and locally quasi-analytic.

Example: If Γ is a weakly branch group, and $\mathfrak{X}$ is the boundary of the tree on which the group acts, then the odometer action of Γ on $\mathfrak{X}$ is inherently not coherent, and not locally quasi-analytic.

Theorem: Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective, coherent and locally quasi-analytic odometer. Let $x \in \mathfrak{X}$. Then there exists an injection

$$\chi_\Phi \colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)$$

For the self-homeomorphisms of $\mathfrak{M}_{\mathcal{P}}$ we obtain:

Theorem: Let $\mathcal{P}$ be a presentation such that the associated odometer $(X_{\mathcal{P}}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Then for $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ we have the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x}) \xrightarrow{\chi_{\Phi}} \mathrm{Comm}(\Gamma)$$

where χ_{Φ} is injective.

Problem: Give conditions for $\sigma_{\mathcal{P},\widehat{x}}$ to be surjective, injective.

- If $\sigma_{\mathcal{P},\widehat{x}}$ is an isomorphism, then calculating $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ is reduced to the algebraic problem of calculating the image of χ_{Φ}.

Example 1: Let $\mathcal{P}$ be coverings of $\mathbb{S}^1$ defined by the chain

$$\Gamma = \mathbb{Z} \supset \Gamma_1 = m_1 \cdot \mathbb{Z} \supset \Gamma_2 = m_1 m_2 \cdot \mathbb{Z} \supset \Gamma_3 = m_1 m_2 m_3 \cdot \mathbb{Z} \supset \cdots$$

where each $m_i > 1$. Set $\vec{m} = \{m_1, m_2, m_3, \ldots\}$, $X_{\vec{m}} = X_{\mathcal{P}}$.

The odometer $(X_{\vec{m}}, \mathbb{Z}, \Phi)$ is free, and $X_{\vec{m}}$ is a profinite completion of $\mathbb{Z}$. Then for any $\widehat{x} \in X_{\vec{m}}$:

- $\sigma_{\mathcal{P},\widehat{x}} \colon \mathrm{Mod}(\mathfrak{M}_{\vec{m}}, \widehat{x}) \to \mathrm{Comm}(X_{\vec{m}}, \mathbb{Z}, \Phi, \widehat{x})$ is an isomorphism
- $\chi_{\Phi} \colon \mathrm{Comm}(X_{\vec{m}}, \mathbb{Z}, \Phi, \widehat{x}) \to \mathrm{Comm}(\mathbb{Z})$ is an inclusion.

Almost everything to say about these solenoids is well-known:

⋆ Kwapisz, *Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua*, **Fund. Math.**, 168(3):251–278, 2001

⋆ Lind & Ward, *Automorphisms of solenoids and p-adic entropy*, **Ergodic Theory Dynam. Systems**, 8:411–419, 1988.

Example 2: Assume that M is a closed manifold, $\Gamma = \pi_1(M, x)$, and $\mathcal{G}$ is a group chain in Γ such that the associated odometer $(X_{\mathcal{G}}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic.

Assume also that M is either:

- an infra-nilmanifold, or
- a Riemannian manifold M with negative sectional curvatures, or
- a Riemannian manifold M of dimension $n \neq 3, 4$ with

non-positive sectional curvatures.

Then for any $\widehat{x} \in X_{\mathcal{P}}$:

- $\sigma_{\mathcal{P},\widehat{x}} \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, \widehat{x})$ is an isomorphism
- $\chi_{\Phi} \colon \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, \widehat{x}) \to \mathrm{Comm}(\Gamma)$ is an inclusion.

⋆ H & Lukina, *Mapping class groups of solenoidal manifolds*, in preparation.

⋆ Davis and Hillman, *The Borel Cconjecture for manifolds with boundary*, arXiv:2501.12509v1.

Example 3: The integer Heisenberg group:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

We denote a 3×3 matrix in $\Gamma_{\mathbb{Z}}$ by the coordinates as (a, b, c). Let $\mathcal{H}$ denote the real Heisenberg group, a, b, c are real numbers.

For distinct primes $p, q \geq 2$, define the self-embedding $\varphi_{p,q} \colon \Gamma \to \Gamma$ by $\varphi(a, b, c) = (pa, qb, pqc)$. Define

$$\Gamma_\ell = \varphi_{p,q}^{\ell}(\Gamma) = \{(p^\ell a, q^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\}$$

which yields a group chain $\mathcal{G}_{p,q}$ in $\Gamma_{\mathbb{Z}}$. The subgroups Γ_ℓ are not normal, and the limit Cantor space $X_{p,q}$ is not a group. The order of the subgroups generated by a and b are invariants.

Let $\mathcal{H}_{p,q}$ be the solenoidal manifold defined by the tower of coverings $\mathcal{H}/\Gamma_\ell$ of $\mathcal{H}/\Gamma_{\mathbb{Z}}$. Then:

$$\mathrm{Mod}(\mathcal{H}_{p,q}, \widehat{x}) \cong \mathbb{Z} \times \mathbb{Z} \times \{\pm 1\} \times \{\pm 1\}$$

Many more examples of nilpotent odometer actions are in:

⋆ H & Lukina, *Type invariants for non-abelian odometers*, **Ergodic Theory Dynamical Systems**, to appear, 2025.

⋆ H & Lukina, *Prime spectrum and dynamics for nilpotent Cantor actions*, **Pacific J. Math.**, 327:107–128, 2023.