

# Foliations

## Thurston Zebras to Cantor Geometries

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# Foliations

In Spring 1974, William Thurston gave a colloquium at Rice University, and spent 20 minutes drawing a picture of a zebra on the blackboard. The message: this is what foliations are about, a very geometric subject!

Later, in graduate courses by

*Franz Kamber, Philippe Tondeur, and also Kuo-Tsai Chen in 1977-79;*

plus Riemannian geometry courses by

*Richard Bishop and Stephanie Alexander;*

more formal approaches emerged.

Still, the definition by Thurston has its attractions, though later his Zebra was replaced by a Chicago Institution:



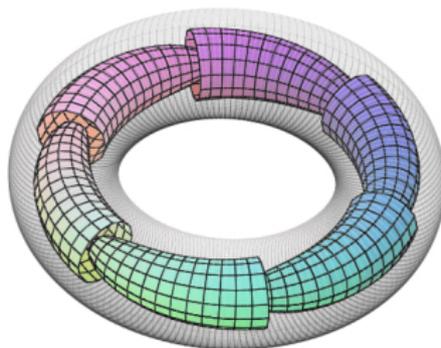
## A Foly Cow!

**Wikipedia:** A foliation is a kind of clothing worn on a manifold, cut from a stripy fabric. On each sufficiently small piece of the manifold, these stripes give the manifold a local product structure. This product structure does not have to be consistent outside local patches: a stripe followed around long enough might return to a different, nearby stripe.

Forty years since the subject was energized by Thurston's works, 1971-75.

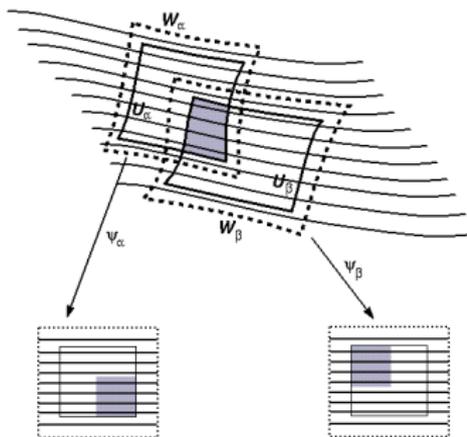
This was one theme at the meeting on *Geometry and Foliations 2013* last September in Tokyo: the contributions of Thurston and the subsequent developments of his ideas in the field. We ignored his famous comment, *that after his work [on foliations], all that was left to do was trivial.*

Historical paper by André Haefliger, available at [foliations.org](http://foliations.org),  
*Naissance des feuilletages, d'Ehresmann-Reeb à Novikov*



## Formal Definition

**Definition:** A foliation  $\mathcal{F}$  of a manifold  $M$  is a “uniform partition” of  $M$  into submanifolds of constant dimension  $p$  and codimension  $q$ . More precisely, a smooth manifold of dimension  $n$  is *foliated* if there is a covering of  $M$  by coordinate charts whose change of coordinate functions preserve the leaves:



A foliation  $\mathcal{F}$  of a compact manifold  $M$  is also . . .

- a local geometric structure on  $M$ , given by a  $\Gamma_{\mathbb{R}^q}$ -cocycle for a “good covering”. (Ehresmann, Haefliger)
- a dynamical system on  $M$  with multi-dimensional time.
- a groupoid  $\mathcal{G}_{\mathcal{F}} \rightarrow M$  with fibers complete manifolds, the holonomy covers of leaves.

Each point of view has advantages and disadvantages.

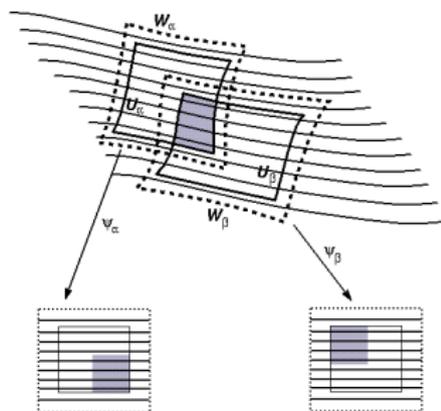
**Problem:** How to distinguish and classify foliations?

Convert to a discrete problem:

## Pseudogroups

A section  $\mathcal{T} \subset M$  for  $\mathcal{F}$  is an embedded submanifold of dimension  $q$  which intersects each leaf of  $\mathcal{F}$  at least once, and always transversally.

Covering by coordinate charts provides local sections, whose union is a global transversal  $\mathcal{T}$ . The transverse holonomy along *leafwise paths* for  $\mathcal{F}$  between points in  $\mathcal{T}$  generates the pseudogroup  $\mathcal{G}_{\mathcal{F}}$ .



# Groupoids

**Definition:** The groupoid of  $\mathcal{G}_{\mathcal{F}}$  is the space of germs

$$\Gamma_{\mathcal{F}} = \{[g]_x \mid g \in \mathcal{G}_{\mathcal{F}} \ \& \ x \in D(g)\} , \ \mathcal{G}_{\mathcal{F}}$$

with source map  $s[g]_x = x$  and range map  $r[g]_x = g(x) = y$ .

**Definition:** The *Haefliger groupoid*  $\Gamma_q^r$  is the collection of all germs of local  $C^r$ -diffeomorphisms between open subsets of  $\mathbb{R}^q$ , endowed with the sheaf topology. The objects of  $\Gamma_q^r$  are points of  $\mathbb{R}^q$ , and morphisms are germs of local  $C^r$ -diffeomorphisms of  $\mathbb{R}^q$ .

The derivative defines a groupoid transformation  $\nu: \Gamma_q^r \rightarrow GL(\mathbb{R}^q)$ .

This space is universal, it is non-Hausdorff, and it is simply huge!

# Haefliger-Thurston classification

$B\Gamma_q^r$  = “classifying space” of (smooth) codimension  $q$ -foliations with transverse differentiability  $C^r$ , introduced by André Haefliger in 1970.

$B\Gamma_q^r \cong \|\Gamma_q^r\| \equiv$  the “semi-simplicial fat realization” of the groupoid  $\Gamma_q^r$ .

For  $r \geq 1$ , there is a natural map  $B\nu: B\Gamma_q^r \rightarrow BGL(\mathbb{R}^q) \cong BO(\mathbb{R}^q)$  classifying the normal bundle to the universal foliation on  $B\Gamma_q^r$ .

**Observation:** [Haefliger] The foliation on  $B\Gamma_q^r$  has a single leaf.

$\Rightarrow$  this is a really strange (non-Hausdorff) space.

- **Theorem:** (Haefliger) Each  $C^r$ -foliation  $\mathcal{F}$  on  $M$  of codimension  $q$  determines a well-defined map  $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^r$  whose homotopy class is uniquely defined by  $\mathcal{F}$ .

*Proof:* Phillips Transversality, Gromov “h-Principle”

- **Theorem:** (Thurston) Each “natural” map  $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^r \times BO_p$  corresponds to a  $C^r$ -foliation  $\mathcal{F}$  of codimension  $q$  on  $M$ , whose concordance class is determined by  $h_{\mathcal{F}}$ .

*Proof:* Draw lots of zebras; that is, very geometric at key steps...

Classification of  $\mathcal{F}$  on  $M \leftrightarrow$  calculate homotopy sets  $[M, B\Gamma_q^r]$

# The Quest

**Problem:** Understand the homotopy type of the mysterious space  $B\Gamma_q^r$ !

Introduce  $F\Gamma_q^r$ , the homotopy fiber of the map  $B\nu: B\Gamma_q^r \rightarrow BO(\mathbb{R}^q)$ .

$F\Gamma_q^r$  is a topological measure of the difference between having a tangent subbundle  $F \subset TM$  of rank  $p$ , and a  $C^r$ -foliation  $\mathcal{F}$  on  $M$  with  $F = T\mathcal{F}$ .

**Theorem:** [Bott Vanishing Theorem, 1968]

$$B\nu^*: H^\ell(BO(\mathbb{R}^q); \mathbb{R}) \longrightarrow H^\ell(B\Gamma_q^r, \mathbb{R})$$

is the zero map for  $r \geq 2$  and  $\ell > 2q$ .

$F\Gamma_q^r$  for  $r \geq 2$  is definitely not trivial. In contrast, there is the amazing

**Theorem:** [Tsuboi, 1989]  $F\Gamma_q^1$  has the weak homotopy type of a point.

## Godbillon-Vey class

**Theorem:** (Godbillon-Vey [1971]) For a  $C^2$ -foliation of codimension  $q$ , there is a cohomology class  $GV(\mathcal{F}) = \Delta(h_1 \cdot c_1^q) \in H^{2q+1}(M; \mathbb{R})$ , whose value depends only on the homotopy class of  $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^2$ .

*Proof:*

- ▶ Assume  $\mathcal{F}$  has oriented normal bundle  $\Rightarrow$  there is a  $q$ -form  $\omega$  with  $\ker(\omega) = T\mathcal{F}$ .
  - ▶ Frobenius Theorem implies the differential  $d\omega = \eta \wedge \omega$  for 1-form  $\eta$ .
  - ▶ Calculation shows  $\eta \wedge (d\eta)^q$  is a closed form.
  - ▶  $GV(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H_{deR}^{2q+1}(M) \cong H^{2q+1}(M; \mathbb{R})$ .
- $\eta$  is the *Reeb modular form* for  $\mathcal{F}$ .
  - $h_1 = [\eta] \in H_{deR}^1(\mathcal{F})$  in leafwise cohomology is well-defined.
  - $c_1 = [d\eta]$  represents the first *Chern class* of the normal bundle to  $\mathcal{F}$ .

**Theorem:** [Roussarie, 1971] The Godbillon-Vey class of the weak-stable foliation  $\mathcal{F}$  on the unit sphere bundle  $M^3$  for a Riemann surface  $\Sigma$  with curvature  $\kappa = -1$  is non-zero.

*Proof:*  $\mathcal{F}$  is also described as the foliation by cosets of the parabolic subgroup of upper triangular matrices in  $SL(\mathbb{R}^2)$ , acting on the unit tangent bundle viewed as the coset space  $M = T^1\Sigma_g = \Gamma \backslash SL(\mathbb{R}^2)$ .

- $\eta$  is the contact form associated to the geodesic flow on  $M$ .
- Calculate  $\eta \wedge d\eta$  is a multiple of the volume form.

**Theorem:** [Thurston, 1971] For each  $\alpha \in \mathbb{R}$ , there exists a foliation  $\mathcal{F}_\alpha$  of codimension-one on  $\mathbb{S}^3$  with  $GV(\mathcal{F}_\alpha) = \alpha \in \mathbb{R} \cong H^3(M; \mathbb{R})$ .

*Proof:* Glue foliations as above together, using clever a observation about flows on unit tangent bundles to Riemann surfaces defined by fundamental domains in  $\mathbb{H}^2$  (the slopes on boundary tori can be controlled...)

## Secondary Classes

**Theorem:** (Bott-Haefliger, Gelfand-Fuks, Kamber-Tondeur [1972])

For each codimension  $q$ , and  $r \geq 2$ , there is a non-trivial space of secondary invariants  $H^*(WO_q)$  and functorial characteristic map whose image contains the Godbillon-Vey class

$$\begin{array}{ccc}
 & H^*(B\Gamma_q^r; \mathbb{R}) & \\
 \tilde{\Delta} \nearrow & & \downarrow h_{\mathcal{F}}^* \\
 H^*(WO_q) & \xrightarrow{\Delta} & H^*(M; \mathbb{R})
 \end{array}$$

The image of  $\Delta$  depends only on the homotopy class of  $h_{\mathcal{F}}: M \rightarrow B\Gamma_q^2$ .

## Cartan approach of Kamber & Tondeur

- $\mathcal{F}$  determines a “partially flat connection”  $\nabla$  on the  $GL(\mathbb{R}^q)$ -bundle of frames orthogonal to the foliation tangent bundle  $T\mathcal{F} \subset TM$ .
- Construct a graded DGA, the “truncated Weil algebra”  $W(\mathfrak{gl}(\mathbb{R}^q), O(q))_{2q}$  with  $H^*(W(\mathfrak{gl}(\mathbb{R}^q), O(q))_{2q}) \cong H^*(WO_q)$ .
- Calculate characteristic map  $H^*(W(\mathfrak{gl}(\mathbb{R}^q), O(q))_{2q}) \rightarrow H^*(B\Gamma_q^2; \mathbb{R})$  using foliations defined by locally homogeneous spaces  $M = \Gamma \backslash G$ , where  $G$  is a semi-simple Lie group of higher rank, with foliation defined by right cosets of a Borel subgroup  $P \subset G$ .
- Key observation is that the non-triviality of the characteristic map follows from non-vanishing of the normal Euler class associated to the action of  $\Gamma$  on the compact quotient space  $X = G/P$ .

*This became the model for almost all examples constructed in the 1970's of foliations which had non-trivial maps to  $B\Gamma_q^2$ .*

## Foliation dynamics

- A *continuous dynamical system* on a compact manifold  $M$  is a flow  $\varphi: M \times \mathbb{R} \rightarrow M$ , where the orbit  $L_x = \{\varphi_t(x) = \varphi(x, t) \mid t \in \mathbb{R}\}$  is thought of as the “time evolution” of the point  $x \in M$ . The trajectories of the points of  $M$  are necessarily *points, circles or lines* immersed in  $M$ , and the study of their aggregate and statistical behavior is the subject of foliation dynamics and ergodic theory for flows.
- *Foliation dynamics, ergodic theory* for foliations of leaf dimension  $\geq 2$ . Replace the concept of time-ordered trajectories with multi-dimensional futures for points. The study of the *dynamics* of  $\mathcal{F}$  asks for properties of the limiting and statistical behavior of the collection of its leaves, and for ergodic theory look for measurable invariants.

## Growth of leaves

An ergodic property of foliations: Riemannian metric on  $M$  yields complete Riemannian metrics on leaves.

For  $x \in M$  and leaf  $L_x$ , consider leafwise balls of radius  $R$ ,

$$B_{\mathcal{F}}(x, R) = \{y \in L_x \mid d_{\mathcal{F}}(x, y) \leq R\}$$

$$Gr(\mathcal{F}, x) = \limsup_{R \rightarrow \infty} \frac{\ln\{\text{Vol}_{L_x}(B_{\mathcal{F}}(x, R))\}}{R} < \infty$$

$Gr(\mathcal{F}, x)$  is measurable orbit invariant.

$L_x$  has *exponential growth* if  $Gr(\mathcal{F}, x) > 0$ .

**Sullivan Conjecture:**[1975] If  $GV(\mathcal{F}) \neq 0$  then the set of leaves with exponential growth is non-empty.

**Theorem:** [Hurder, Jour. Diff. Geom. 1984]

If  $GV(\mathcal{F}) \neq 0$  for  $\mathcal{F}$  a codimension  $q \geq 1$  foliation, then the set of leaves with exponential growth has positive Lebesgue measure.

*Idea of Proof:* Uses joint work with James Heitsch, inspired by work of Gerard Duminy, and *cocycle tempering* à la *Pesin Theory*.

The *Reeb modular class*  $[\eta] \in H_{deR}^1(\mathcal{F})$  is an ergodic property of  $\mathcal{F}$ , and the operator norm of  $[\eta]$  has estimate,

$$\|[\eta]\| \sim \int_M Gr(\mathcal{F}, x) dvol_M(x)$$

More refined ergodic invariants, use cocycles in spirit of Zimmer program.

Derivative map gives measurable cocycle  $D\nu: \Gamma_{\mathcal{F}} \rightarrow GL(\mathbb{R}^n)$

For any parabolic subgroup  $P \subset GL(\mathbb{R}^n)$  get an action on homogeneous space  $X_P = GL(\mathbb{R}^n)/P$

**Theorem:**[Hurder & Katok, Annals of Math 1987]

- $[D\nu]_e \in H_{erg}^1(\Gamma_{\mathcal{F}}; X_P)$  is ergodic invariant
- Equivalence relation defined by  $\mathcal{F}$  measurably amenable  $\Rightarrow [D\nu]_e = 0$ .
- If  $[D\nu]_e = 0$  then many of the secondary classes vanish.

**Conclusion:**

$B\Gamma_q^2$  Foliation Classification  $\Leftrightarrow$  Foliation Dynamics & Ergodic Properties.

# Foliation Dynamics & Ergodic Theory

Major theme of work in the field since 1980:

- Random walks on leaves, leafwise harmonic measures  
[Garnett, 1980] , [Ghys, 1995], [Candel, 2003], [Deroin & Klepsyn, 2007],  
[Deroin, Klepsyn & Navas, 2007]
- Foliation geometric entropy [Ghys, Langevin & Walczak, 1988]
- Lyapunov spectrum and foliation dynamics [Hurder, Walczak, 1988]
- Rigidity of Anosov systems [Hurder & Katok, 1990], [Hurder, 1992],  
[Katok, Spatzier, et al, 1993 onwards]
- Regularity of weak stable foliations [Hasselblatt, 1992 onwards],  
[Pugh, Shub & Wilkinson, 1997 onwards], etc

## Mather-Thurston Theorem

$\text{Diff}_c(\mathbb{R}^q) =$  compactly supported  $C^\infty$ -diffeomorphisms of  $\mathbb{R}^q$ .

$\text{Diff}_c^d(\mathbb{R}^q) = \text{Diff}_c(\mathbb{R}^q)$  with discrete topology. HUGE!

*Most amazing* “unknown” theorem:

**Theorem:** [Mather & Thurston, 1974] There is a natural map  $\sigma: \text{Diff}_c^d(\mathbb{R}^q) \rightarrow \Omega^q F\Gamma_q^\infty$  which induces isomorphisms on homology

$$H^*(\Sigma^q \text{Diff}_c^d(\mathbb{R}^q); \mathbb{Z}) \cong H^*(F\Gamma_q^\infty)$$

$F\Gamma_q^\infty$  is the homotopy fiber of the universal map  $B\nu: B\Gamma_q^r \rightarrow BO(\mathbb{R}^q)$ .

**Question:** What are the cycles in  $H^*(\Sigma^q \text{Diff}_c^d(\mathbb{R}^q); \mathbb{Z})$ ?

Propose to construct cycles using dynamical systems!

## Lipschitz Cantor Geometries

Only one “Cantor set”, call it  $\mathfrak{X}$ , but has many metrics:

$d_{\mathfrak{X}}$  and  $d'_{\mathfrak{X}}$  are *Lipschitz equivalent*, if they satisfy a Lipschitz condition:

$$C^{-1} \cdot d_{\mathfrak{X}}(x, y) \leq d'_{\mathfrak{X}}(x, y) \leq C \cdot d_{\mathfrak{X}}(x, y) \quad \text{for all } x, y \in \mathfrak{X}$$

**Definition:** A compactly generated pseudogroup  $\mathcal{G}_{\mathfrak{X}} = \langle g_1, \dots, g_k \rangle$  acting on a Cantor set  $\mathfrak{X}$  is *Lipschitz* with respect to a metric  $d_{\mathfrak{X}}$  on  $\mathfrak{X}$ , if there exists  $C \geq 1$  such that for each  $1 \leq i \leq k$  then

$$C^{-1} \cdot d_{\mathfrak{X}}(w, w') \leq d_{\mathfrak{X}}(h_i(w), h_i(w')) \leq C \cdot d_{\mathfrak{X}}(w, w'), \quad \forall w, w' \in \text{Dom}(g_i).$$

**Definition:** *Lipschitz Cantor Geometry* investigates properties of *minimal Lipschitz pseudogroups actions* on metric Cantor sets  $(\mathfrak{X}, d_{\mathfrak{X}})$ .

Work with coauthors Alex Clark and Olga Lukina during past 5 years has studied properties of *matchbox manifolds*, which are *foliated spaces* with transversal Cantor model  $\mathfrak{X}$ , and holonomy is pseudogroup action on  $\mathfrak{X}$ .

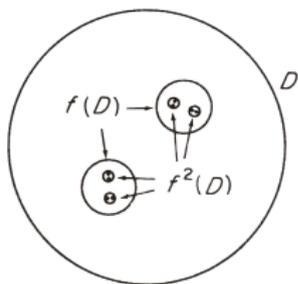
**Definition:**  $C^r$  Cantor Geometry,  $r \geq 1$ , investigates the properties of Cantor actions satisfying:

- ▶  $\mathfrak{X} \subset \mathbb{R}^q$  is an embedding,
- ▶  $\mathcal{G}_{\mathfrak{X}} = \langle g_1, \dots, g_k \rangle$  is generated by restrictions of  $C^r$ -diffeomorphisms on some open neighborhood of  $\mathfrak{X} \subset U \subset \mathbb{R}^q$ .

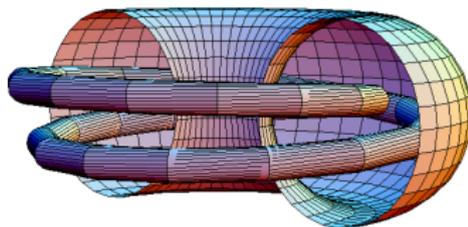
### Why study these objects?

- Interesting in their own right – invariant sets for  $C^2$ -dynamical systems.
- For  $r \geq 2$ , examples which yield non-trivial classes in  $H^*(F\Gamma_q^r; \mathbb{R})$ , not in the image of the usual secondary characteristic maps.

# Smoothly embedded solenoids



*Cantor Set  $\mathfrak{X}$*



*Solenoid in  $\mathbb{R}^3$*

The holonomy along leaves of  $\mathcal{F}$  “twist” to the points of  $\mathfrak{X}$ .

Pseudogroup maps on  $\mathfrak{X}$  extend to open neighborhoods of  $\mathfrak{X}$ .

Solenoids naturally arise as hyperbolic attractors of smooth flows.  
 [Smale, 1967], [Gambaudo, Sullivan, Tresser, 1990 – onwards]

**Theorem:** [Clark & Hurder, 2011] For  $p \geq 1$  and  $q \geq 2n$ , there exists commuting diffeomorphisms  $\varphi_i: \mathbb{S}^q \rightarrow \mathbb{S}^q$ ,  $1 \leq i \leq p$ , so that the suspension of the induced action  $\mathbb{Z}^p$  on  $\mathbb{S}^q$  yields a smooth foliation  $\mathcal{F}$  with solenoidal minimal set  $\mathcal{S}$ , such that:

- The leaves of  $\mathcal{F}$  restricted to  $\mathcal{S}$  are all isometric to  $\mathbb{R}^p$
- The isotropy groups of periodic orbits form a profinite series

$$\cdots \Gamma_i \subset \cdots \Gamma_1 \subset \Gamma_0 = \mathbb{Z}^n$$

- $\mathfrak{X} \cong \varprojlim (\Gamma_0/\Gamma_i)$  is an “adic-completion” of  $\mathbb{Z}^p$ .
- Every open neighborhood of  $\mathfrak{X}$  contains periodic open domains for the action of  $\mathbb{Z}^p$  on  $\mathbb{S}^q$

**Question:** What is relation to the quest?

**Theorem:** [Heitsch's Thesis, 1970]

The Bott Vanishing Theorem is false for  $\mathbb{Z}$  coefficients.

**Example:** Let  $(\mathbb{Z}/p\mathbb{Z})^m$  act on  $\mathbb{D}^{2m}$  via rotations  $\{\varphi_1, \dots, \varphi_m\}$  with period  $p$  on each of the  $m$ -factors of  $\mathbb{D}^2$ .

Form the suspension flat bundle  $\mathbb{E} = \mathbb{S}^\infty \times \mathbb{D}^{2m}/\varphi$ . Then

$$\nu_{\mathbb{E}}^*: H^*(BSO(2m); \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(B\Gamma_{2m}^+; \mathbb{Z}/p\mathbb{Z}) \rightarrow H^*(\mathbb{E}; \mathbb{Z}/p\mathbb{Z})$$

is injective. Let  $p \rightarrow \infty$ , then observe associated  $\lim_1$ -terms not zero, so Pontrjagin classes do not vanish in  $H^*(B\Gamma_q^2; \mathbb{Z})$ .

For any open neighborhood  $Z \subset U$  of the solenoidal minimal set, there are infinite sequences of “Heitsch Examples” embedded in  $\mathcal{F}|U$ .

The restricted foliation  $\mathcal{F}|U$  yields a map  $h_Z: B\Gamma_{U|\mathcal{F}} \rightarrow B\Gamma_q^\infty$ .

**Theorem:** Let  $\mathcal{S}$  be the solenoidal minimal set above. Then the homotopy class of  $h_Z: B\Gamma_{U|\mathcal{F}} \rightarrow B\Gamma_q^2$  is non-trivial.

Moreover, for  $\ell > q/2$  there is a natural surjection

$$h_Z^*: H^{4\ell-1}(B\Gamma_q^2; \mathbb{R}) \rightarrow \mathcal{H}^{4\ell-1}(\mathcal{S}, \mathcal{F}; \mathbb{R})$$

**Proof:** The Cheeger-Simons classes derived from  $H^*(BSO(q); \mathbb{R})$  are in the image of  $h_Z^* \dots$

These solenoidal cycles are just the simplest type of iterated braids.

There are many more  $C^2$  Cantor geometries...

After 40 years, the study of the classification problem for foliations has produced wide-ranging new techniques, and new perspectives on traditional and more recent topics in dynamics.

As for the quest to “find”  $B\Gamma_q^r$  – not even close to understanding this mysterious space!

Thank you for your attention!

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