The shape of the minimal set of the Kuperberg plug

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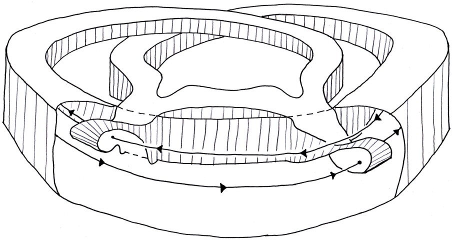
Motivation

Theorem (K. Kuperberg, 1994)

Let M be a closed 3-manifold. Then M admits a C^∞, or even real analytic, non-vanishing vector field with no periodic orbits.



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For a generic Kuperberg plug, the minimal set has topological dimension 2 and is stratified, having a 1-dimensional stratum that accumulates on the 2-dimensional stratum.

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The minimal set has unstable shape.

Definition (Stable shape)

A compact set Σ *has stable shape if there exists a shape approximation* $\mathfrak{U} = \{U_\ell \mid \ell = 1, 2, \ldots\}$ *such that each inclusion* $\iota\colon U_{\ell+1} \hookrightarrow U_\ell$ *induces a homotopy equivalence, and* U_1 *has the homotopy type of a finite polyhedron.*

Construction

$$\Sigma = \overline{\mathfrak{M}} = \overline{\cup_{i=0}^{\infty}\mathfrak{M}_i}.$$

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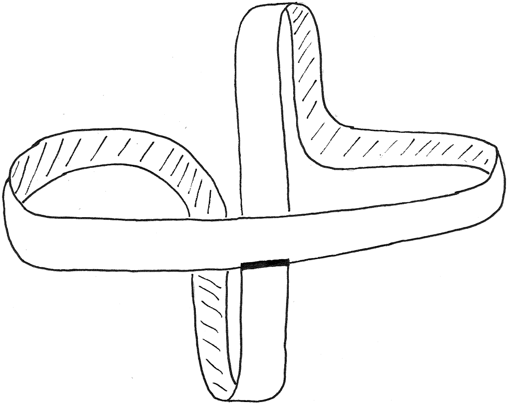
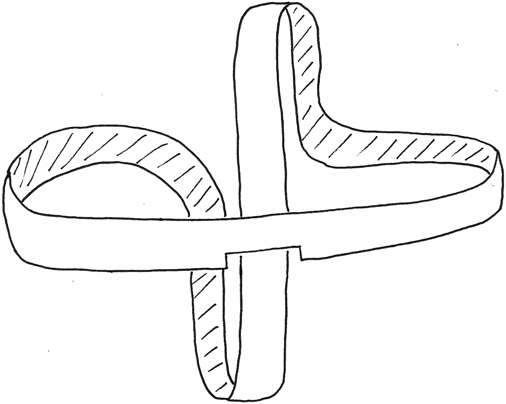


Construction

$$\mathfrak{M}_0$$



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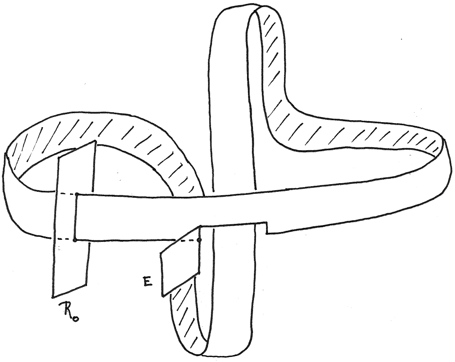
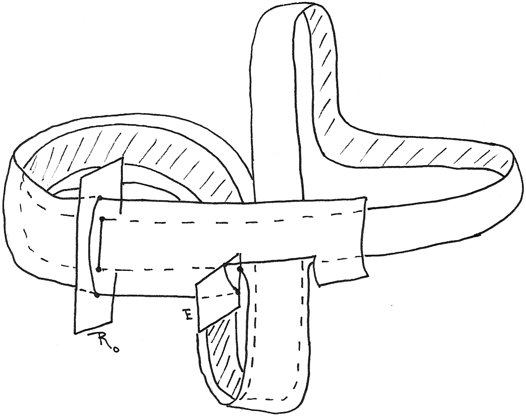


Set $f : E \to \mathbb{R}$ the distance of a point to the special point $\partial R \cap E$.

Construction



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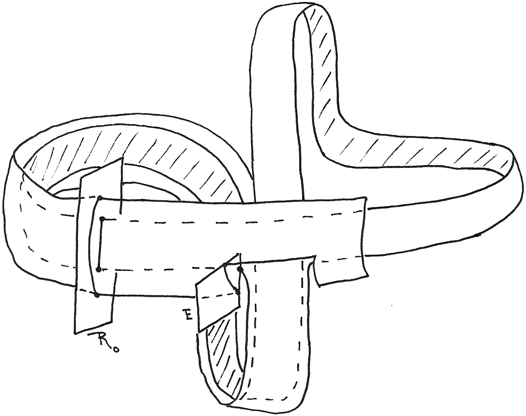

Construction of $\mathfrak{M}_k$ from $\mathfrak{M}_{k-1}$ for $k \geq 2$.

Unstable shape

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Theorem

$\Sigma \subset \mathbb{R}^3$ *has unstable shape.*

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Proposition (Strategy of proof)

Let U_n be a shape approximation of Σ such that for every $k \geq 2$:

- *the rank of $H_1(U_k; \mathbb{Z}) > 2^{k-1}$;*
- *there exists $\ell > k$ such that the rank of the image $H_1(U_\ell; \mathbb{Z}) \to H_1(U_K; \mathbb{Z})$ is 2.*

Assume that for any shape approximation V_n the rank of $H_1(V_n; \mathbb{Z})$ is greater than 2, then Σ has unstable shape.

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Proof.

Assume that Σ has stable shape and let V_n be a good shape approximation. Set $n_0 > 2$ to be the rank of the image $H_1(V_k) \to H_1(V_\ell)$, for every $k \geq \ell$ and ℓ big enough. Take U_n as in the statement.

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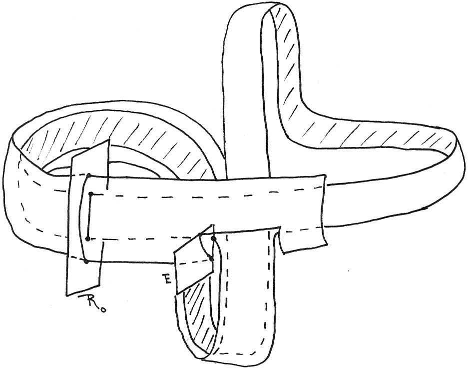
For ℓ big enough, $\exists n_1, k_1, n_2, k_2$ such that $V_{n_2} \subset U_{k_2} \subset V_{n_1} \subset U_{k_1} \subset V_\ell$ and

$$H_1(V_{n_2}) \to H_1(U_{k_2}) \to H_1(V_{n_1}) \to H_1(U_{k_1}) \to H_1(V_\ell).$$

Shape approximation

Nesting property

The set $\mathfrak{M}_k \setminus \mathfrak{M}_{k-1}$ admits a one sided closed neighborhood F_k that contains $\mathfrak{M} \setminus \mathfrak{M}_{k-1}$.



Shape approximation

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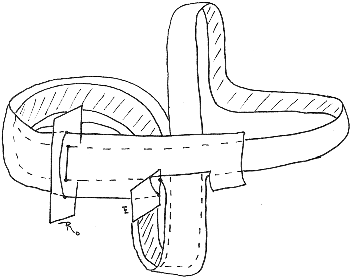
$$\mathfrak{N}_1 = (\mathfrak{M}_0, \delta_0) \cup F_1.$$



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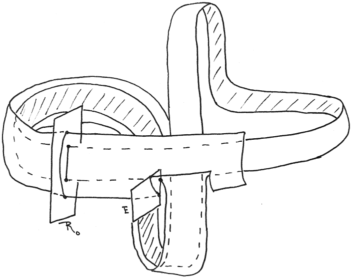

In general

$$\mathfrak{N}_k = (\mathfrak{M}_{k-1}, \delta_{k-1}) \cup F_k.$$