Solenoidal minimal sets in foliations

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Reeb–Thurston–Stowe Stability Theorems

Let $\mathcal{F}$ be a C^r-foliation of a smooth compact manifold M, for $r \geq 1$.

Theorem: (Reeb [1952]) Let L be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $\pi_1(L, x) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

Theorem: (Thurston [1974]) Let L be a compact leaf of a codimension one foliation $\mathcal{F}$ such that $H^1(L, \mathbb{R}) = 0$. Then there exists an open saturated neighborhood $L \subset U$ such that $\mathcal{F} \mid U$ is a product foliation.

Theorem: (Stowe [1983]) Let L be a compact leaf of a codimension q foliation $\mathcal{F}$ such that $H^1(L, \mathbb{V}) = 0$ for all flat finite-dimensional vector bundles associated to a representation of $\pi_1(L, x)$. Then there exists an open saturated neighborhood $L \subset U$ such that if $\mathcal{F}'$ is a sufficiently C^1 close to $\mathcal{F}$, then $\mathcal{F}' \mid U$ is a product foliation.

Instability of leaves

Suppose that L is a compact leaf with $H^1(L, \mathbb{R}) \neq 0$. It is trivial to construct foliations with L as a leaf, such that every leaf L' which intersects an open neighborhood $L \subset U$ is necessarily non-compact.

Problem: Suppose that L is a compact leaf with $H^1(L, \mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. Then what can be said about the behavior of a foliation $\mathcal{F}'$ which is C^1 close to $\mathcal{F}$?

This question arose from a related problem posed by Alex Clark:

Problem: Given a solenoid $\mathcal{S}$ with p-dimensional leaves, when does there exists a smooth foliation $\mathcal{F}$ of a compact manifold such that $\mathcal{S}$ is a minimal set for $\mathcal{F}$?

Embedded solenoids

Theorem: (Clark & Hurder [2006]) $\mathcal{F}$ is a codimension q C^1-foliation. Let L be a compact leaf with $H^1(L, \mathbb{R}) \neq 0$, and $L \subset U$ is a saturated open neighborhood for which $\mathcal{F} \mid U$ is a product foliation. Then there exists $\mathcal{F}'$ arbitrarily C^1 close to $\mathcal{F}$ such that U is saturated for $\mathcal{F}'$, $\mathcal{F}$ and $\mathcal{F}'$ agree outside of U, and $\mathcal{F}' \mid U$ has a compact solenoidal minimal set $\mathbf{K} \subset U$, where

$$\mathbf{K} = \varprojlim \{f_n \colon L_{n+1} \to L_n\}$$

where each L_n is a covering of L.

Remark: The proof uses new constructions in foliation theory, and seems of interest for the questions it raises about perturbations of foliations.

Flat bundles

Choose a basepoint $x \in L$, and set $\Gamma = \pi_1(L, x)$.

Γ acts on the right as deck transformations of the universal cover $\widetilde{L} \to L$.

Let $\rho \colon \Gamma \to \mathbf{SO(q)}$ be an orthogonal representation.

Γ acts on the left as isometries of $\mathbb{R}^q$ by $\gamma \cdot \vec{v} = \rho(\gamma)\vec{v}$.

Define a flat $\mathbb{R}^q$-bundle with holonomy ρ by

$$\mathbb{E}^q_\rho = (\widetilde{L} \times \mathbb{R}^q)/(\widetilde{y} \cdot \gamma, \vec{v}) \sim (\widetilde{y}, \gamma \cdot \vec{v}) \longrightarrow L$$

The most familiar example is for $L = \mathbb{S}^1$ and $\Gamma = \pi_1(\mathbb{S}^1, x) = \mathbb{Z} \to \mathbf{SO(2)}$. Then $\mathbb{E}^2_\rho$ is the flat vector bundle over $\mathbb{S}^1$ with the foliation by lines of slope $\rho(1) = \exp(2\pi\sqrt{-1}\,\alpha)$.

In general, the bundle $\mathbb{E}^q_\rho \to L$ need not be a product vector bundle.

Trivializing flat bundles

Proposition: Suppose that there exists a 1-parameter family of representations $\rho_t \colon \Gamma \to \mathbf{SO(q)}$ such that ρ_0 is the trivial map, and $\rho_1 = \rho$, then ρ_t canonically defines a vector bundle map $\mathbb{E}^q_\rho \cong L \times \mathbb{R}^q$.

Proof: The family of representations defines a family of flat bundles $\mathbb{E}^q_{\rho_t}$ over the product space $L \times [0,1]$. This defines an isotopy between the bundles $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$, which induces a bundle isomorphism between them. The initial bundle $\mathbb{E}^q_{\rho_0}$ is a product, hence the same holds for $\mathbb{E}^q_{\rho_1}$.

In the case of the example above over $\mathbb{S}^1$ the product structure can be written down explicitly.

The key point is that the bundle isomorphism between $\mathbb{E}^q_{\rho_0}$ and $\mathbb{E}^q_{\rho_1}$ depends smoothly on the path ρ_t.

Abelian representations

$k =$ the greatest integer such that $2k \leq q$.

$\mathbb{T}^k \subset \mathbf{SO(q)}$ a maximal embedded k-torus.

$\xi = (\xi_1, \ldots, \xi_k) \colon \Gamma \to \mathbb{R}^k$ a representation. Define

$$\begin{aligned} \rho_t^{\xi} \colon \Gamma &\to \mathbf{SO(q)} \\ \rho_t^{\xi}(\gamma) &= [\exp(2\pi t\sqrt{-1}\xi_1(\gamma)), \ldots, \exp(2\pi t\sqrt{-1}\xi_k(\gamma))] \end{aligned}$$

$$\begin{aligned} \mathbb{D}_{\epsilon}^{q} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 < \epsilon\} \subset \mathbb{R}^q \\ \mathbb{B}_{\epsilon}^{q} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 \leq \epsilon\} \subset \mathbb{R}^q \\ \mathbb{S}_{\epsilon}^{q-1} &= \{(z_1, \ldots, z_q) \mid z_1^2 + \cdots z_q^2 = \epsilon\} \subset \mathbb{R}^q \end{aligned}$$

Realizing abelian representations

Proposition: $\xi\colon \Gamma \to \mathbb{R}^k$ defines a flat bundle foliation $\mathcal{F}_\xi$ of $L \times \mathbb{S}^{q-1}$ whose leaves cover L. Moreover, if the image of ξ is contained in the rational points $\mathbb{Q}^k \subset \mathbb{R}^k$, then all leaves of $\mathcal{F}_\xi$ are compact.

Proof: ρ_t^ξ is an isotopy from ξ to the trivial representation. $\square$

The idea is that if we take a path $\lambda\colon [0,\epsilon] \to \mathbf{Rep}(\Gamma, \mathbf{SO(q)})$ of such representations, then this will yield a foliation $\mathcal{F}_\lambda$ of $L \times \mathbb{D}^q_\epsilon$ whose restriction to the spherical fiber $L \times \mathbb{S}^{q-1}_s$ is $\mathcal{F}_{\lambda(s)}$, for $0 \leq s < \epsilon$.

The basic plug

In the main theorem, we can assume that $U = L \times \mathbb{D}^q_\epsilon$.

Fix a non-trivial representation $\xi_1 \colon \Gamma \to \mathbb{Q}^k$ which exists as $H^1(L, \mathbb{R}) \neq 0$.

Let $0 < \epsilon/2 < \epsilon_1 < \epsilon$, and set $\epsilon_1' = (\epsilon_1 + \epsilon)/2$. Choose a monotone decreasing smooth function $\mu_1 \colon [0, \epsilon] \to [0, 1]$ such that

$$\mu_1(s) = \begin{cases} 1 & \text{if } \ 0 \leq s \leq \epsilon_1, \\ 0 & \text{if } \ \epsilon_1' \leq s \leq \epsilon \end{cases}$$

Set $\rho^{\xi_1}_{1,s} = \rho^{\mu_1(s)\xi_1} \colon \Gamma \to \mathbf{SO(q)}$. Use this family of representations to define a foliation $\mathcal{F}_1$ of $N_1 = L \times \mathbb{D}^q_\epsilon$.

Note that $\mathcal{F}_1$ is the product foliation outside of $L \times \mathbb{S}^{q-1}_{\epsilon_1'}$, and has all leaves compact in $L \times \mathbb{B}^q_{\epsilon_1}$.

Iterating the plug

Let L_1 be a generic leaf of $\mathcal{F}_1$ contained in $L \times \mathbb{S}^{q-1}_{\epsilon_1/2}$.

By construction, $L_1 \longrightarrow L$ is the compact covering associated to the kernel $\Gamma_1 \subset \Gamma$ of the homomorphism $\rho^{\xi_1} \colon \Gamma \longrightarrow \mathbf{SO(q)}$.

Next choose $0 < \epsilon_2 < \epsilon$ sufficiently small so that $\mathcal{F}_1$ restricted to the ϵ_2-disk bundle N_2 about L_1 is a product foliation.

We now repeat the construction: choose a non-trivial map $\xi_2 \colon \Gamma_1 \longrightarrow \mathbb{Q}^k$ and maps μ_2 as before.

Iterating the plug 2

Iterate for all $n \geq 2$. This yields:

A descending sequence of subgroups $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$

An increasing of open saturated subsets

$$V_n = L \times \mathbb{D}^q_\epsilon - L_{n+1} \times \mathbb{B}^q_{\epsilon_{n+1}}$$

with a foliation $\mathcal{F}'_n$ which has leaves of increasingly high order as coverings, corresponding to the subgroups Γ_n

Note that $V_n \subset V_{n+1}$, hence $\mathbf{K}_n = L \times \mathbb{D}^q_\epsilon - V_n$, forms a nested sequence of compact sets, $\mathbf{K}_{n+1} \subset \mathbf{K}_n$.

The perturbation $\mathcal{F}'$

Proposition: If the maps ξ_n are suitably chosen (ie the images of the generators of Γ are approach 0 in $\mathbb{Q}^k$ sufficiently rapidly) then:

1. the foliations $\mathcal{F}'_n$ converge to a C^r-foliation $\mathcal{F}'$ of $L \times \mathbb{D}^q_\epsilon$.
2. $\mathbf{K} = \cap_{n=1}^{\infty} \mathbf{K}_n$ is a saturated compact set.
3. $\mathcal{F}' \mid \mathbf{K}$ is a solenoid.

Problem: Is there a classification for the solenoids which arise in this way?

Look for an answer to more general formulation.

Pseudogroups & Groupoids

Every foliation admits a discrete model by choosing a section $\mathcal{T} \subset M$, an embedded submanifold of dimension q which intersects each leaf of $\mathcal{F}$ at least once, and always transversally. The holonomy of $\mathcal{F}$ yields a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$.

Definition: A pseudogroup of transformations $\mathcal{G}$ of $\mathcal{T}$ is *compactly generated* if there is

- a relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}$ such that $\langle \Gamma \rangle = \mathcal{G}|\mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}$ with $\overline{D(g)} \subset D(\widetilde{g}_i)$.

Definition: The groupoid of $\mathcal{G}$ is the space of germs

$$\Gamma_{\mathcal{G}} = \{[g]_x \mid g \in \mathcal{G} \;\&\; x \in D(g)\} \;,\; \Gamma_{\mathcal{F}} = \Gamma_{\mathcal{G}_{\mathcal{F}}}$$

with source map $s[g]_x = x$ and range map $r[g]_x = g(x) = y$.

Derivative cocycle

Assume $(\mathcal{G}, \mathcal{T})$ is a compactly generated pseudogroup, and $\mathcal{T}$ has a uniform Riemannian metric. Choose a uniformly bounded, Borel trivialization, $T\mathcal{T} \cong \mathcal{T} \times \mathbb{R}^q$, $T_x\mathcal{T} \cong_x \mathbb{R}^q$ for all $x \in \mathcal{T}$.

Definition: The normal cocycle $D\varphi \colon \Gamma_{\mathcal{G}} \times \mathcal{T} \to \mathbf{GL}(\mathbb{R}^{\mathbf{q}})$ is defined by

$$D\varphi[g]_x = D_x g \colon T_x\mathcal{T} \cong_x \mathbb{R}^q \to T_y\mathcal{T} \cong_y \mathbb{R}^q$$

which satisfies the cocycle law

$$D\varphi([h]_y \circ [g]_x) = D\varphi[h]_y \cdot D\varphi[g]_x$$

Twist invariant

Proposition: Given a closed saturated subset $\mathbf{K} \subset M$,

$$[D\varphi \mid \mathbf{K}] \in H^1(\mathcal{G}_{\mathcal{F}} \mid \mathbf{K}; \mathbf{GL}(\mathbb{R}^{\mathbf{q}})) \cong H^1_{\mathcal{F}}(\mathbf{K}; \mathbf{GL}(\mathbb{R}^{\mathbf{q}}))$$

is an invariant of $\mathcal{F} \mid \mathbf{K}$.

Theorem: Let $\mathcal{F}'$ be a foliation with solenoidal minimal set $\mathcal{S}$ as above. Then $[D\varphi \mid \mathcal{S}] \in H^1_{\mathcal{F}'}(\mathcal{S}; \mathbf{GL}(\mathbb{R}^{\mathbf{q}}))$ is non-trivial, and measures the "asymptotic twisting of the lamination".

Problem: Calculate $H^1_{\mathcal{F}'}(\mathcal{S}; \mathbf{GL}(\mathbb{R}^{\mathbf{q}}))$ for a solenoid.

Asymptotic expansion

Definition: For $g \in \Gamma_{\mathcal{G}}$, the word length $\|[g]\|_x$ of the germ $[g]_x$ of g at x is the least n such that

$$[g]_x = [g_{i_1}^{\pm 1} \circ \cdots \circ g_{i_n}^{\pm 1}]_x$$

Word length is a measure of the "time" required to get from one point on an orbit to another.

Definition: The transverse expansion rate function at x is

$$\lambda(\mathcal{G}, n, x) = \max_{\|[g]\|_x \leq n} \frac{\ln\left(\max\{\|D_x g\|, \|D_y g^{-1}\|\}\right)}{\|[g]\|_x} \;\geq\; 0$$

Invariant measures

Definition: The asymptotic transverse growth rate at x is

$$\lambda(\mathcal{G}, x) = \limsup_{n \to \infty} \ \lambda(\mathcal{G}, n, x) \ \geq \ 0$$

This is essentially the maximum Lyapunov exponent for $\mathcal{G}$ at x.

Theorem: Let $\mathbf{K} \subset M$ be a compact saturated subset such that $\lambda(\mathcal{G}, x) = 0$ for all $x \in \mathbf{K} \cap \mathcal{T}$. Then $\mathcal{F} \mid \mathbf{K}$ has a holonomy invariant transverse measure supported on $\mathbf{K}$.

Distal foliations

Definition: A foliation $\mathcal{F}$ is distal if its pseudogroup $(\mathcal{G}_{\mathcal{F}}, \mathcal{T})$ is distal: that is, for all $x \neq y \in \mathcal{T}$ there exists $\epsilon_{x,y} > 0$ such that

$$d_{\mathcal{T}}(g(x), g(y)) \geq \epsilon_{x,y} \text{ for all } g \in \mathcal{G}_{\mathcal{F}}$$

Definition: A foliation is said to be *compact* if all leaves of $\mathcal{F}$ are compact submanifolds.

Remark: All compact foliations are distal.

Theorem: If $\mathcal{F}$ is distal and transversally $\mathbf{C}^{1+\alpha}$ for some $\alpha > 0$, then $\lambda(\mathcal{G}, x) = 0$ for all $x \in \mathcal{T}$.

As an application, this gives another proof that a minimal set $\mathbf{K} \subset M$ has a holonomy invariant transverse measure supported on $\mathbf{K}$.

Problem: Is there a smooth structure theory for solenoids in distal foliations?