DYNAMICS OF A PIECEWISE ROTATION

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Abstract. We investigate the dynamics of systems generalizing interval exchanges to planar mappings. Unlike interval exchanges and translations, our mappings, despite the lack of hyperbolicity, exhibit many features of attractors. The main result states that for a certain class of noninvertible piecewise isometries, orbits visiting both atoms infinitely often must accumulates on the boundaries of the attractor consisting of two maximal invariant discs $D_0 \cup D_1$ fixed by T. The key new idea is a dynamical and geometric observation about the monotonic behavior of orbits of a certain first-return map. Our model emerges as the local map for other piecewise isometries and can be the basis for the construction of more complicated molecular attractors.

1. Introduction

In this paper, we study systems generalizing well known and studied interval exchanges to a class of Euclidean two-dimensional piecewise isometries. Piecewise isometries appear in a variety of contexts and have been recently extensively studied as interval exchanges [2, 4, 6, 10, 12, 17, 20], interval translations [5], rectangular exchanges [9], polygonal and polyhedron exchanges [1, 8] and pseudogroup systems of rotations [16]. Piecewise isometric maps appear naturally in billiards [3], theory of foliations [18] and tilings.

This work is partially motivated by the recent work of Boshernitzan and Kornfeld [5] investigating the noninvertible piecewise isometries in dimension one. The authors studied the behavior of the convergence of decreasing sequence of sets : $X, T\,X, T^2X, \ldots$ for interval translation maps $T : X \to X$.

We investigate noninvertible piecewise isometries in dimension two with the particular interest on the maximal invariant sets and $\omega-$limit sets. Unlike in [5], the induced isometries T_0 and T_1 of our system $T : X \to X$ are not translations but rotations. The partition $\mathcal{P}$ consists of two atoms: P_0 - the open left halfplane and P_1 - the closed right halfplane. The atom P_i is rotated by angle α_i about the point S_i. The dynamics of T depends on the angles of rotations and the position of the centers of rotation S_0 and S_1 (Figures 1, 4 and 5).

Our central theorem describes the dynamics of T when the centers of rotations S_0 and S_1 are fixed by T and the line S_0S_1 is perpendicular to the discontinuity line for T and it can be summarized as follows:

Theorem. *(Theorem 1) For irrational choice of angles of rotation, all orbits accumulate in the two maximal invariant discs $D_0 \cup D_1$ fixed by T (Figure 2). There exist orbits visiting both atoms infinitely often, and these orbits accumulate on the boundaries of $D_0 \cup D_1$. Moreover, $D_0 \cup D_1 = \overline{\cap_{n>0} T^n\,Y}$ for all bounded sets $Y \supset D_0 \cup D_1$.*

The key idea in the proof is a dynamical and geometric observation about the monotonic behavior of orbits of the first-return map to one of the atoms.

Our result has a number of interesting applications. It allows us, for example, to determine the invariant measures of the system leading to the computation the topological entropy of the

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continuous extension $\hat{T}$. Finally, the theorem can be used to describe the local dynamics of some other classes of piecewise isometries since the piecewise rotation T emerges a local model for other piecewise rotations.

The structure of this article is as follows. In section 2, we define the coding map $\sigma : X \to \Omega$ as the main descriptive tool in the study of piecewise isometries. We also construct a continuous extension space for piecewise isometries. The main result with a number of consequences is stated in section 3. The proof is included in section 4. In section 5, we point out that our main techniques used in the proof of Theorem 1 work for some other cases. In section 6, we illustrate an example of a perturbed system for which the studied model emerges as a local map and gives rise to more complex attractors and symbolic dynamics. We conclude the article in section 7 by addressing some related questions and proposing investigation leading to the classification of all piecewise isometries.

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2. Piecewise Isometries and their Coding

In this section, we define piecewise isometries and their natural coding of orbits. We also show that coding gives rise to a space on which an extension of piecewise isometries can be viewed as a continuous mapping. Propositions proved in this section shall be used in the discussion of the main result.

Even though the notion of a piecewise isometry is quite general, since in this article we are primarily interested in Euclidean piecewise isometries, we restrict our definition to Euclidean systems.

Definition 1. Let $X \subset \mathbb{R}^n$ and let $\{T_0, \ldots, T_{r-1}\}$ $(r > 0)$ be a finite collection of isometries of $\mathbb{R}^n$. A transformation $T : X \to X$ (not necessarily invertible) is called a *piecewise isometry* with partition $\mathcal{P} = \{P_0, \ldots, P_{r-1}\}$ and *induced isometries* $\{T_0, \ldots, T_{r-1}\}$ if (i) $T|_{P_i} = T_i|_{P_i}$ for every $i = 0, \ldots, r-1$, (ii) $P_i \cap P_j = \emptyset$ for $i \neq j$, and (iii) $\bigcup_{0 \leq j \leq r-1} P_j = X$.

Some cases of invertible piecewise isometries have been studied in the literature. The most basic ones are interval exchange transformations. These systems were first defined by Keane [12] and studied extensively for example in [2, 4, 6, 10, 12, 17, 21]. An *interval exchange map* is a surjective piecewise isometry acting on a right-open interval whose partition consists of smaller right-open intervals and whose induced isometries are translations.

An important result in the subject is that even though there are examples of minimal nonuniquely ergodic interval exchanges [13, 15], the set of these systems has measure zero in a certain parameter space. The unique ergodicity of typical interval exchanges was proved independently by Masur [17] and Veech [21] and later by Boshernitzan [4], Kerkhoff [14] and Rees [19].

Invertible piecewise isometries in dimension two or higher were studied for example in [8, 9]. Haller studies *rectangular exchanges* (the space and atoms are rectangles) and it particular he determines when typical families of rectangular exchanges are ergodic. The work of Haydn and Gutkin, is written in a more general context, investigates the entropy and symbolic complexity of *polygonal and polyhedral exchanges.*

2.1. Coding of orbits.

The partition $\mathcal{P}$ of the space X gives rise to a natural coding σ of orbits. The coding σ encodes the forward orbit of a point $x \in X$ by recording the indices of atoms visited by the orbit.

Definition 2. Let $\sigma : X \to \Omega = \{0, \ldots, r-1\}^{\mathbb{N}}$ be the coding map defined by $\sigma(x) = w_0 w_1 \ldots$ where $T^k x \in P_{w_k}$ and $w_k \in \{0, \ldots, r-1\}$.

The mapping σ naturally induces a refinement of the partition $\mathcal{P}$, which is the *entropy or coding* partition Σ of X. Σ is induced by the equivalence relation $x \sim y$ if and only if $\sigma(x) = \sigma(y)$. One can also check that $\Sigma = \bigvee_{m=0}^{\infty} T^{-m}\mathcal{P}$. After [11], the elements of Σ will be called *cells* .

Proposition 1. *Suppose that the partition $\mathcal{P}$ consists of convex sets. Then every cell is a convex set.*

Proof. Take some element $U \in \Sigma$ whose coding $\sigma(U) = \sigma_0\sigma_1\sigma_2 \ldots$. Observe that U is precisely the set of all points $x \in X$ such that $x \in P_{\sigma_0}$ and such that $T_{\sigma_i} \ldots T_{\sigma_1} T_{\sigma_0}\, x \in P_{\sigma_{i+1}}$ for every $i \geq 0$. Therefore,

$$U = P_{\sigma_0} \,\cap\, (T_{\sigma_0}^{-1} P_{\sigma_1}) \,\cap\, (T_{\sigma_0}^{-1} T_{\sigma_1}^{-1} P_{\sigma_2}) \,\cap\, (T_{\sigma_0}^{-1} T_{\sigma_1}^{-1} T_{\sigma_2}^{-1} P_{\sigma_3}) \,\cap\, \ldots$$

is the intersection of convex sets, and hence U is convex. □

The coding map σ conjugates T with the (one-sided) *shift map* $S : \Omega_r \to \Omega_r$, $S(w_0 w_1 w_2 \ldots) = w_1 w_2 \ldots$:

$$\begin{array}{ccc} X & \xrightarrow{\ T\ } & X \\ \sigma \big\downarrow & & \big\downarrow \sigma \\ \Omega_r & \xrightarrow{\ S\ } & \Omega_r \end{array} \tag{1}$$

The shift map $S : \Omega_r \to \Omega_r$ is continuous with respect to the topology generated by the metric $d_{\Omega_r} : \Omega_r \times \Omega_r \to \mathbb{R}^+ \cup 0$, $d_{\Omega_r}(\sigma_1, \sigma_2) = \sum_{i=0}^{\infty} \frac{|\sigma_1(i) - \sigma_2(i)|}{r^i}$.

2.2. **The graph construction.** One of the key applications of the coding is the construction of a larger space on which an extension T_G of T is continuous. Constructions of such spaces can be achieved by "separating" cells. The construction we present here is a bit different from the topological constructions in [5, 8, 12] as it is applicable to noninvertible maps and uses the graph of the coding.

Let $G = \{(x, \sigma(x)),\ x \in X\}$ be the graph of $\sigma : X \to \Omega$ topologized by the product metric $d_G : G \times G \to \mathbb{R}^+ \cup 0$, $d_G(\,(x_1, \sigma(x_1)), (x_2, \sigma(x_2))\,) = \max\{d(x_1, x_2), d_{\Omega_r}(\sigma(x_1), \sigma(x_2))\}$, where d is the Euclidean metric on X. Then

Proposition 2. *The extension map $T_G : G \to G$, $(x, \sigma(x)) \to (T\,x, \sigma(T\,x))$ is continuous.*

Proof of Proposition 2. Let $x_0 \in P_i$. For all $x \notin P_i$, $d_G((x_0, \sigma(x_0)), (x, \sigma(x))) \geq d_{\Omega_r}(\sigma(x_0), \sigma(x)) \geq 1$. Therefore, in the limit statement below, we may also assume that $x \in P_i$.

$$\begin{aligned}
\lim_{(x,\sigma(x)) \to (x_0,\sigma(x_0))} T_G\,(x, \sigma(x)) &= \lim_{(x,\sigma(x)) \to (x_0,\sigma(x_0))} (T\,x, \sigma(T\,x)) && \text{by definition of } T_G \\
&= \lim_{(x,\sigma(x)) \to (x_0,\sigma(x_0))} (T_i\,x, S(\sigma(x))) && \text{since } x \in P_i \text{ and by (1)} \\
&= (T_i\,x_0, S(\sigma(x_0))) && \text{by continuity of } T_i \text{ and } S \\
&= (T_i\,x_0, \sigma(T\,x_0)) = T_G(x_0, \sigma(x_0)) && \text{by (1) and definition of } T_G.
\end{aligned}$$

□

The existence of a continuous extended space for piecewise isometries is crucial as it allows us to apply many results from continuous dynamical systems. For example the following useful corollary can be used in the estimates of the symbolic complexity of the system discussed briefly in section 7.

Corollary 1. Let $\overline{G_T} = \hat{X}$. Let $\hat{T}: \ \hat{X} \to \hat{X}$ be the continuous extension of $T_G\colon G_T \to G_T$. Then the natural projection $\Pi_{\Omega_r}|_{\hat{X}}: \ \hat{X} \to \overline{\sigma(X)}$ is a continuous, surjective mapping and $S|_{\overline{\sigma(X)}}: \overline{\sigma(X)} \to \overline{\sigma(X)}$ is a topological factor map of $\hat{T}$. In particular, the following diagram commutes.

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{T}} & \hat{X} \\ \Pi_{\Omega_r}\big\downarrow & & \big\downarrow \Pi_{\Omega_r} \\ \overline{\sigma(X)} & \xrightarrow{S} & \overline{\sigma(X)} \end{array}$$

3. Main Result and Corollaries

In this section, we study a particular subclass of piecewise isometries. The generating map is the piecewise rotation T with two atoms separated by a discontinuity line. The induced isometries which are rotations about points S_0 and S_1 fixed by T and the line $S_0 S_1$ is perpendicular to the discontinuity line (Figure 2 and Figure 4, case 1). The description of some other cases is postponed to sections 4 and 5.

Our main result describes the dynamics of orbits visiting both atoms infinitely often. The dynamics of orbits which are eventually trapped in one of the atoms is especially simple as it is equivalent to the dynamics of the rotation on a circle.

Theorem 1 (8-attractor map). *Let $T_0, T_1 : \mathbb{C} \to \mathbb{C}$ the rotations by angles $\alpha_0, \alpha_1 \in (0, 2\pi)$ about the centers: $S_0 = -r_0$ and $S_1 = r_1$, respectively ($r_0, r_1 \in \mathbb{R}^+$). Let $T : \mathbb{C} \to \mathbb{C}$ be the piecewise isometry with induced isometries $\{T_0, T_1\}$ and the partition $\{P_0, P_1\}$ of $\mathbb{C}$ into the open left halfplane and the closed right halfplane (Figures 1, and 3). Let $\mathcal{J} \subset \mathbb{C}$ be the set of points with non-eventually constant codings. Then:*

1. *If at least one of the rotations T_0 or T_1 is rational (${T_0}^s = id$ or ${T_1}^s = id$ for some s), then $\mathcal{J} = \emptyset$.*

Otherwise, if both rotations T_0 and T_1 are irrational, then $\mathcal{J} \neq \emptyset$, and:

2. *The accumulation set of every orbit beginning in $\mathcal{J}$ is the union of the two circles: $\partial D_0 \cup \partial D_1$ where $D_0 = \overline{B(S_0, r_0)}$ and $D_1 = \overline{B(S_1, r_1)}$.*
3. *The coding $\sigma(x)$ of every point $x \in \mathcal{J}$ is irrational (not eventually periodic) and has the property that the gaps between consecutive switches between zeros and ones tend to infinity.*
4. *Let $Y \subset \mathbb{C}$ be a bounded set containing $D_0 \cup D_1$. Then,*

$$\bigcap_{n \geq 0} T^n Y = D_0 \cup D_1 - \{T_0^n 0 : \ n > 0\}.$$

Piecewise rotations $T \in \mathcal{T}$ described in the above result will be further called an *8-attractor maps* since an orbit which visits both atoms infinitely often eventually traces the shape of the figure eight.

Remark 1. The Housdorff dimension and measure of $\mathcal{J}$ is not known.

Theorem 1 has a number of interesting applications some of which we shall now discuss. Another application to more complex piecewise isometries is presented in section 6. In order to state the first corollary, we define piecewise isometric attractors by modifying standard definitions (this modification is necessary as the map T is not continuous).

Definition 3. A *piecewise isometric attractor* for the map $T : X \to X$ is a compact set $A \subset X$ such that for some open set $U \supset X$, $T(U) \subset U$ and $Z \cup \bigcap_{n>0} T^n U = A$ where Z is a countable set.

Let $Y = B(S_0, R_0) \cup B(S_1, R_1)$ (where $R_0 > r_0$ and $R_1 > r_1$ and $R_0^2 - r_0^2 = R_1^2 - r_1^2$) be an open neighborhood of $D_0 \cup D_1$. Then $T(Y) \subset Y$ and Conclusion 4 implies that

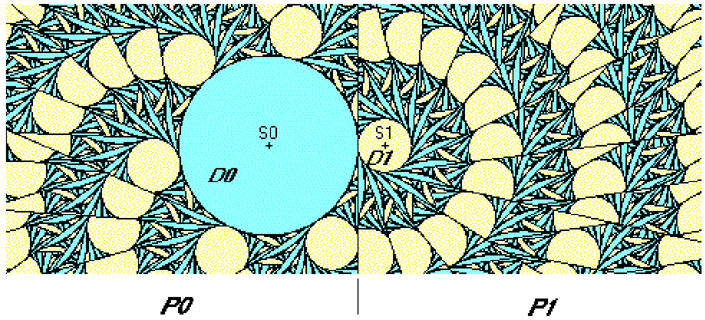

FIGURE 1. Illustration of the 8-attractor. The two centered discs D_0 and D_1 are fixed by the map T and they constitute a piecewise isometric attractor. The shaded regions are the cells with eventually constant coding. Darker cells are ultimately trapped in the disc D_0 and lighter cells are trapped in D_1. Points visiting both atoms infinitely often are contained the back region.

Corollary 2. (**Attractor**) For irrational rotations T_0 and T_1, $D_0 \cup D_1$ is a piecewise isometric attractor for the piecewise rotation T.

Another interesting conclusion is that if two different points u and v have the same codes, then all points in the line segment $\overline{uv}$ have the same eventually constant coding. This follows from the following corollary.

Corollary 3 (**Injectivity of coding).** The coding map $\sigma : \mathbb{C} \to \Omega$ restricted to the set of all points with non-eventually constant codes $\mathcal{J}$ is injective.

Note that it follows explicitly from Theorem 1 that every point with non-eventually constant code has an irrational code.

Proof of Corollary 3. Suppose that the points u and v have the same non-eventually constant coding $\omega \in \Omega$. Let $\{n_m\}$ be the subsequence of all positive integers such that $T^{n_m} u \in P_1$. The atoms are the halfplanes, from Proposition 1 it thus follows that the line segment $\overline{u\,v}$ is contained in one cell, hence it follows the same coding pattern and $\overline{T^{n_m} u\, T^{n_m} v} \subset P_1$ for every m. The codings of the points in $\overline{T^{n_m} u\, T^{n_m} v}$ are (the same and) non-eventually constant, thus $\overline{T^{n_m} u\, T^{n_m} v}$ is disjoint from the disc D_1. Since $\mathrm{d}(u, v) = \mathrm{d}(T^{n_m} u,\, T^{n_m} v)$, from elementary geometry it follows that

$$\max(\mathrm{d}(T^{n_m} u, S_1),\, \mathrm{d}(T^{n_m} v, S_1)) \geq \sqrt{r_1^2 + \frac{\mathrm{d}(u,v)^2}{4}}. \tag{2}$$

Conclusion 2 implies that $T^{n_m} u \to \partial D_1$ and $T^{n_m} v \to \partial D_1$ as $m \to \infty$, thus (2) gives us $u = v$. □

The subsequent corollary describes invariant measures of T restricted to a compact set. (The action of T can be restricted to the union of two closed discs: $Y = \overline{B(S_0, R_0) \cup B(S_1, R_1)}$ for any choice of radii $R_0 > r_0$ and $R_1 > r_1$ satisfying $R_0^2 - r_0^2 = R_1^2 - r_1^2$ since $TY \subset Y$). Determining invariant measures can be used in the estimates of the symbolic complexity and the entropy of the system. We shall briefly discuss this topic in section 7.

Note that any invariant measure is always supported on the maximal invariant set $M = \bigcap_{n>0} f^n X$ of the map $f : X \to X$, for if γ is an invariant measure with respect to f, then $\gamma(M) = \lim_{n\to\infty} \gamma(f^n X) = \gamma(f^{-n}(f^n X)) = \gamma(X)$. Hence, Theorem 1 (Conclusion 4) yields:

Corollary 4 (**Invariant measures).** If both isometries T_0 and T_1 are irrational rotations, then every non-atomic Borel probability invariant measure for $T|_Y : Y \to Y$, where $Y \supset D_0 \cup D_1$ is a compact set such that $T(Y) \subset Y$, is supported on the union of the discs $D_0 \cup D_1$.

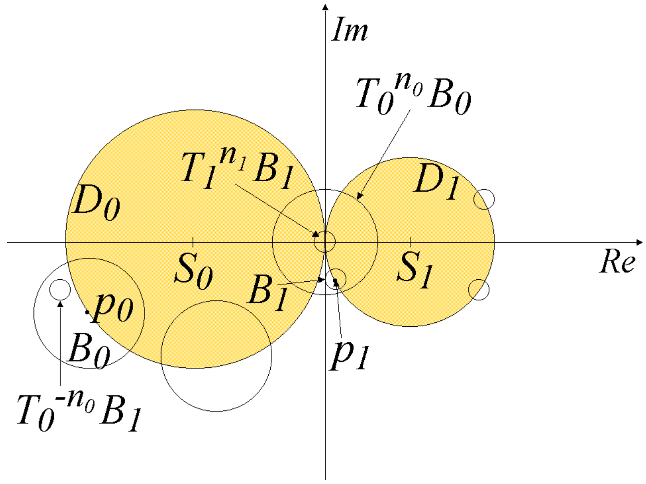

FIGURE 2. Illustration to the proof that $\mathcal{J} \neq \emptyset$.

In particular, since T leaves D_0 and D_1 invariant, and T restricted to either D_0 or D_1 is an irrational rotation, the invariant measures for T are equivalent to the Lebesgue measures on discs.

The last corollary describes the recurrent set. Recall that the *recurrent set* $R(f)$ for the map $f : X \to X$ is the set of all points $x \in X$ such that for every neighborhood U of x there is an integer $n > 0$ such that $f^n(x) \subset U$.

Corollary 5 (Recurrent set). The recurrent set $R(T)$ for T is the set of points with constant coding. In particular, If T_0 and T_1 are irrational rotations then $\mathbb{R}(T) = D_0 \cup D_1 - \{T_0^{-n}0 : \ n > 0\}$ (since T is not continuous the recurrent set is not closed).

Proof. Certainly, all points with constant codings are recurrent. On the other hand, suppose that $x \in R(x)$. If the coding of x were irrational then its orbit would accumulate only on the circles $\partial D_0 \cup \partial D_1$ (Conclusion 2) and thus $x \in \partial D_0 \cup \partial D_1$. However, $\partial D_0 \cup \partial D_1$ contains only points with constant or eventually constant codings, thus the coding of x cannot be irrational. Therefore, by Theorem 1 the coding of x must be eventually constant. It follows that $\overline{\{T^n x : \ n > 0\}}$ is either a circle or a finite set. In both cases since x must be in the same atom P_i as $\overline{\{T^n x : \ n > 0\}}$. Moreover, the orbit of x never escapes from P_i. It follows that the coding of x is constant. □

4. Proof of Theorem 1

Proof. The proof consists of two parts. Part 1 is the detailed analysis to the forward orbit of a point whose orbit visits both atoms infinitely often. Part 1 proves all conclusions of the theorem with the exception of conclusion 4 about maximal invariant set. Conclusion 4 is proved in part 2 where we investigate the backward orbit of a point outside $D_1 \cup D_2$.

4.1. **Part 1.** First we show that if both T_0 and T_1 are irrational rotations, then there exist points with non-eventually constant coding. In step 2, we show that the orbit $\{T^n\, x,\, n \geq 0\}$ converges to the union of two circles: $\partial B(S_0, d_\infty) \cup \partial B(S_1, e_\infty)$ ($d_\infty \geq r_0$ and $e_\infty \geq r_1$). In step 3, we argue that if the circles $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$ intersected at two different points, then x would have eventually periodic, non-constant code. In step 4, we show that this is not possible, hence the circles $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$ are tangent, that is $d_\infty = r_0$ and $e_\infty = r_1$. Finally, in step 5, we draw the main conclusions. We argue that the induced isometries must be both irrational rotations, all points of the circles ∂D_0 and ∂D_1 are the accumulation points of the orbit of x, and that the coding of x is irrational.

Step 1. We first show that $\mathcal{J} \neq \emptyset$.

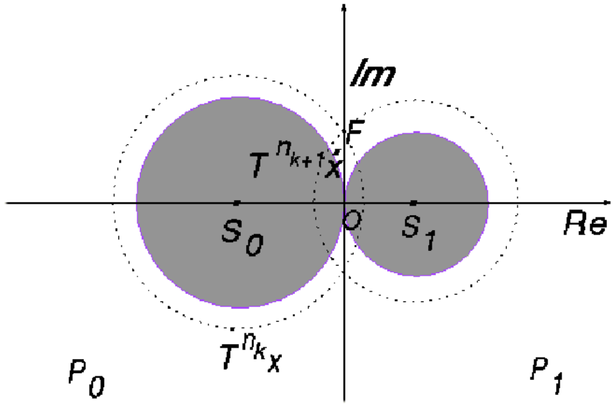

FIGURE 3. Illustration to the proof of Theorem 1

Pick a point $p_0 \in \partial D_0$ such that $T_0^{n_0} p_0 = 0$ for some $n_0 > 0$. Then choose $\epsilon > 0$ small enough so that the closed ball $B_0 = B(p_0, \epsilon_0)$ has the property that $T_0^k B_0 \subset \mathrm{Int} P_0$ for $0 \le k < n_0$. Observe (Figure 1) that $T_0^{n_0} B_0$ contains a neighborhood of a point $p_1 \in \partial D_1$ such that $T_0^{n_1} p_1 = 0$ for some $n_1 > 0$. Choose $\epsilon_1 > 0$ small enough so that the closed ball $B_1 = B(p_1, \epsilon_1)$ lies in $\mathrm{Int} P_0$ and $T_1^{k_1} B_1 \subset P_1$ for $0 \le k < n_1$. Then choose a point $p_2 \in \partial D_0 \cap T_1^{n_1}(\mathrm{Int} B_1)$ such that $T_0^{n_2} p_2 = 0$ for some $n_2 > 0$ and choose $\epsilon_2 > 0$ small enough so that the closed ball $B_2 = B(p_2, \epsilon_2)$ has the property that $T_0^k B_2 \subset \mathrm{Int} P_0$ for $0 \le k < n_2$. Proceeding in this fashion, we shall obtain an infinite sequence of closed balls $B_0, B_1, B_2, \ldots$ such that the even numbered balls are in $\mathrm{Int} P_0$, the odd numbered balls are in $\mathrm{Int} P_1$. Moreover,

$$B_0 \supset T_0^{-n_0} B_1 \supset T_0^{-n_0} T_1^{-n_1} B_2 \supset T_0^{-n_0} T_1^{-n_1} T_0^{-n_2} B_3 \subset \ldots .$$

Since the nested balls are closed, their intersection is non-empty; it must lie in a single cell with non-eventually constant coding.

Step 2. We show that the orbit $\{T^n x, n \ge 0\}$ converges to the union of two circles: $\partial B(S_0, d_\infty) \cup \partial B(S_1, e_\infty)$ ($d_\infty \ge r_0$ and $e_\infty \ge r_1$). Suppose that the point $x \in \mathbb{C}$ does not have an eventually constant coding. Let $\{n_k\}$ be the subsequence of all positive integers such that $T^{n_k} x \in P_0$. Let $d_k = \mathrm{d}(T^{n_k} x, S_0)$. Obviously, $d_k = d_{k+1}$ for all k such that $n_{k+1} = n_k + 1$. On the other hand, for all exiting times from P_0, that is for all k such that $n_{k+1} > n_k + 1$, we show that

$$d_k^2 - d_{k+1}^2 \ge 2\, \epsilon_{k+1}(r_0 + r_1) \text{ where } \epsilon_{k+1} = |\mathrm{Re}\ (T^{n_{k+1}} x)| > 0. \tag{3}$$

The orbit of x lies outside of the disc D_0 (otherwise the orbit of x would be trapped in one of the atoms), so the circle $\partial B(S_0, d_k)$ intersects the non-negative half of the imaginary axis at some point F (Figure 3). The iterates: $T^{n_k+1} x, \ldots, T^{n_{k+1}} x$ are in the inside of the circle $\partial B(S_1, \mathrm{d}(S_1, F))$, so

$$\mathrm{d}(T^{n_{k+1}} x, S_1)^2 \le \mathrm{d}(S_1, F)^2 = \mathrm{d}(F, S_0)^2 - \mathrm{d}(S_0, O)^2 + \mathrm{d}(O, S_1)^2 = d_k^2 - r_0^2 + r_1^2.$$

Using the above inequality, we estimate the distance d_{k+1} of the iterate $T^{n_{k+1}} x$ from S_0:

$$d_{k+1}^2 = (r_0 + \mathrm{Re}\ (T^{n_{k+1}} x))^2 + (\mathrm{Im}(T^{n_{k+1}} x))^2 = (r_0 - \epsilon_{k+1})^2 + \mathrm{d}(T^{n_{k+1}} x, S_1)^2 - (r_1 + \epsilon_{k+1})^2$$

$$\le (r_0 - \epsilon_{k+1})^2 + (d_k^2 - r_0^2 + r_1^2) - (r_1 + \epsilon_{k+1})^2 = d_k^2 - 2\epsilon_{k+1}(r_0 + r_1). \tag{4}$$

Inequality (3) follows from (4).

As a nonincreasing sequence, $\{d_k\}$ has a limit, say $d_k \to d_\infty$ as $k \to \infty$. Let $\{n_m\}$ be the subsequence of all positive integers such that $T^{n_m}\, x \in P_1$. Let $e_m = \mathrm{d}(T^{n_m}\, x, S_1)$. Similarly, we show that $\{e_m\}$ is a nonincreasing sequence, thus has a limit, say $e_m \to e_\infty$.

Let $\{k_l\}$ be the subsequence of the exiting times from P_0, that is: $\{k_l\} = \{k\colon\, n_{k+1} > n_k + 1\}$. From inequality (3), we conclude that $\epsilon_{k_l+1} \to 0$ as $l \to \infty$. Passing to the limit over the subsequence $\{k_l\}$ in the first line of the estimate (4), we obtain that

$$\lim_{l\to\infty} \mathrm{d}(T^{n_{k_l+1}}\, x, S_1)^2 = d_\infty^2 + r_1^2 - r_0^2. \tag{5}$$

On the other hand, $T^{n_{k_l+1}}\, x$ is an entering P_0 iterate, hence $T^{n_{k_l+1}-1}\, x \in P_1$ and $n_{k_l+1}-1 \in \{n_m\}$. We therefore obtain

$$\lim_{l\to\infty} \mathrm{d}(T^{n_{k_l+1}}\, x, S_1)^2 = \lim_{l\to\infty} \mathrm{d}(T^{n_{k_l+1}-1}\, x, S_1)^2 = e_\infty^2. \tag{6}$$

Combining identities (5) and (6), we obtain $d_\infty^2 - r_0^2 = e_\infty^2 - r_1^2$, which means that the limit circles: $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$ intersect in the imaginary axis.

Let O' and O'' be the intersection points of the circles $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$. In the next two steps we show that $O' = O''$ that is the limit circles $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$ are tangent to each other.

Step 3. Suppose otherwise, assume that $O' \neq O''$. We conclude that the orbit of x must have an eventually periodic, nonconstant code, and that x is an eventually periodic point. The entering P_0 iterates $T^{n_{k_l+1}}\, x$ must converge to $\{O' \cup O''\}$ because $\epsilon_{k_l+1} = |\mathrm{Re}\, (T^{n_{k_l+1}}\, x)| \to 0$ as $l \to \infty$ (inequality (3)). (Similarly, the entering P_1 iterates must converge to $\{O' \cup O''\}$.) Let $T^{n_s}\, x$ be the subsequence of those entering P_0 iterates converging to one of the points in $O' \cup O''$, say O'. Let $T_0^a O'$ $(a > 0)$ be the first point on the forward T_0-orbit of O' which enters the closed atom P_1. (Note that such an a must exist. For otherwise, since the circle $\partial B(S_0, d_\infty)$ contains the arc in P_1, T_0 would have to be a rational rotation. In this case, for some $a > 0$, $T_0^a O' = O' \in P_1$.) Then $T_0^a O'$ lies on $\partial B(S_0, d_\infty) \cap P_1 = \partial B(S_0, d_\infty) \cap \overline{B(S_1, e_\infty)}$. The point $T_0^a O'$ cannot lie in the interior of $B(S_1, e_\infty)$, for otherwise for sufficiently close to O' iterates $T^{n_s} x$, $T^a(T^{n_s}) = T_0^a(T^{n_s} x) \in B(S_1, e_\infty)$. This would mean that the forward $T-$orbit of x enters the interior of $B(S_1, e_\infty)$, contrary to the choice of $B(S_1, e_\infty)$. Also, $T_0^a O' \neq O'$, for otherwise $T_0^a = Id$, and $T^a(T^{n_s} x) = T^{n_s} x$ for iterates T^{n_s} sufficiently close to O'. This would mean that the orbit of x is eventually trapped in P_0, contrary to the assumption that the orbit of x visits both atoms infinitely often.

$$T_0^a\, O' = O''. \tag{7}$$

By the choice of the number a for the iterates $T^{n_s}\, x$ sufficiently close to O', $T^a(T^{n_s}\, x) = T_0^a(T^{n_s}\, x)$. Moreover, $T_0^a(T^{n_s}\, x) \in P_1$, for otherwise would have the inequality of angles $|\langle T^{n_s}\, x\, S_0\, T_0^a(T^{n_s}\, x)| > |\langle O' S_0 O''|$, contrary to equation (7). Thus O'' is an accumulation point of entering P_1 iterates of x. Let $T_1^b O''$ $(b > 0)$ be the first point on the forward T_1-orbit of O'' which enters the closed closure of the atom P_0. By applying a similar argument to the above, we conclude that

$$T_1^b\, O'' = O'. \tag{8}$$

Finally, observe that for entering P_0 iterates $T^{n_s}\, x$ sufficiently close to O', the subsequent entering P_0 iterates $T^{n_s}\, x$ enter in an arbitrarily small neighborhood of O'; hence

$$T^{n_{s+1}} x = T^{a+b}(T^{n_s}\, x) = T_1^b T_0^a (T^{n_s}\, x)$$

for iterates $T^{n_s}\, x$ sufficiently close to O' and thus x is an eventually periodic point, and it has an eventually periodic coding.

Step 4. Since O' is fixed by $T_1^b T_0^a$ (equations (7) and (8)), and since $T_1^b T_0^a$ is an isometry, it follows that the iterates $T^{n_s}\, x$ cannot converge to O', a contradiction with our initial choice of the subsequence $T^{n_s}\, x$. We have thus shown that the assumption that $O' \neq O''$ is false. Therefore, the orbit $\{T^n\, x,\, n \geq 0\}$ converges to the union of the circles $\partial D_0 \cup \partial D_1$.

Step 5. In this step, we draw conclusions 1, 2, and 3. We show that the accumulation set of the orbit of every point is the union of the circles $\partial D_0 \cup \partial D_1$ (conclusion 2), both of the induced isometries have infinite order (conclusion 1) and the coding of x is irrational (conclusion 3). For every ϵ, the entering P_0 iterates, as well as the entering P_1 iterates, must eventually lie in the $\epsilon-$neighborhood of the imaginary axis. This means that the distance between entering P_0 iterate and the subsequent entering P_1 iterate, must tend to zero. In particular, the induced isometry T_0 must be an irrational rotation and the time that the orbit spends in the atom P_0 must tend to infinity. Hence the circle ∂D_0 is the accumulation set of the orbit intersected with P_0. Similarly, we show that T_1 must be an irrational rotation and that the circle ∂D_1 is the accumulation set of the orbit intersected with P_1.

Finally, since the coding of x is not eventually constant, then, as the orbit of that point clusters on the circles: $\partial D_0 \cup \partial D_1$, it has to spend arbitrary long time in each atom. Hence the coding of that point cannot be eventually periodic.

4.2. **Part 2.** We now prove conclusion 4 of Theorem 1. The main idea in the proof below is apply the reasoning from part 1 to backward orbits of points, and then to use the fact that Y is bounded.

Given a map $f : Y \to Y$ let us define a *backward chain* initiated by $x_0 \in Y$ and generated by f to be a sequence $\{x_{-k}\}$ such that $f\,x_{-k} = x_{-k+1}$ for $k \geq 1$ ($\{x_{-k}\}$ can be either infinite or finite). Let $A = D_0 \cup D_1$. We first claim that it is enough to show that for every $x_0 \in Y - A$, every backward chain initiated by x_0 is finite.

The set $\bigcap_{n\geq 0} f^n\,Y$ is the set of all points with arbitrarily long backward chains. However, in the case of the piecewise isometry T, a stronger conclusion can be drawn. Since T is at most two-to-one, every point with arbitrarily long backward chain, initiates an infinite backward chain. Hence $\bigcap_{n\geq 0} T^n\,Y$ is exactly the set of all points initiating at least one infinite backward chain. On the other hand, note that with the exception of $\{T_0^n 0 :\ n > 0\}$, every point in A initiates an infinite backward chain. Hence, it is sufficient to show that every backward chain initiated by $x_0 \in Y - A$ is finite.

Suppose otherwise, let $x_{-1}, x_{-2}, \ldots,$ be an infinite backward chain beginning at x_0, that is $T\,x_{-n} = x_{-n+1}$ for $n \geq 1$, and $x_0 = x$. Let us denote: $T^{-n}\,x = x_{-n} \in Y - A$ (we slightly abuse the notation here since T is not $1-1$). We now repeat the ideas presented in part 1 of the proof Theorem 1 and the fact that the space Y is bounded.

First, observe that the backward chain $x_{-1}, x_{-2}, \ldots,$ alternates between the atoms P_0 and P_1. If not, suppose that for every $n \geq m_0$, $x_{-n} \in P_0$. Then $x_{-m_0} \in \bigcap_{i\geq 0} T_0^i\,P_0 \subset B(S_0, r_0)$. Since the (forward) orbit of every point in $B(S_0, r_0)$ is contained in $B(S_0, r_0) \subset A$, in particular $x_0 \in A$, a contradiction with our assumption.

We now extend the definitions of subsequences form part 1 in order to apply the arguments from steps 3, and 4. As in step 2, let $\{n_k\}$ be the subsequence of all integers (positive and negative) such that $T^{n_k}\,x \in P_0$. Let $\{n_m\}$ be the subsequence of all integers such that $T^{n_m}\,x \in P_1$. Since the backward chain $x_0, x_{-1}, x_{-2}, \ldots$ alternates between P_0 and in P_1, both sequences $\{n_k\}$ and $\{n_m\}$ are bi-infinite. Let $d_k = \mathrm{d}(T^{n_k}\,x, S_0)$ and $e_m = \mathrm{d}(T^{n_m}\,x, S_1)$ be defined as in part 1. The argument from step 2 shows that sequences d_k and $\{e_m\}$ are nondecreasing (as $k \to -\infty$ and $m \to -\infty$). Since Y is bounded, they must have limits:

$$\lim_{k\to-\infty} d_k = d_\infty \quad \text{and} \quad \lim_{m\to-\infty} e_m = e_\infty.$$

The backward chain $x_{-1}, x_{-2}, \ldots,$ has to converge to the union of the limit circles: $\partial B(S_0, d_\infty)$ and $\partial B(S_1, e_\infty)$ where $d_\infty > r_0$ and $e_\infty > r_1$ (since $e_0 > r_0$ and $e_1 > r_1$).

We now arrived at the same point in the proof as in the beginning of step 3 in part 1. Hence, in order to arrive at a contradiction, we repeat steps 3, and 4, taking all the limits as indexes tend to minus infinity rather then plus infinity. □

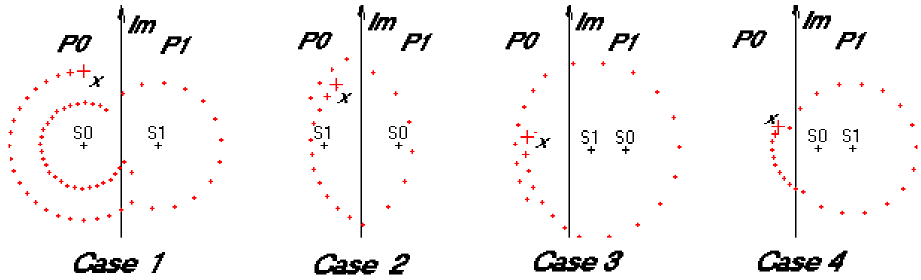

FIGURE 4. Four different cases of the position of centers of rotations on the real axis resulting in different dynamics.

5. OTHER CASES

In this section, we illustrate that the techniques used in the proof of Theorem 1 can be applied to some other cases of piecewise rotations with two atoms.

Theorem 1 describes the case of the dynamics of the piecewise rotation T for which there are two fixed points S_0 and S_1 lying on the real axis Re $S_0 < 0 <$ Re S_1. If one of these points is moved slightly away from the real axis, the conclusions of Theorem 1 fail to hold as for some choices of parameters defining T, there exist new periodic discs (see Figure 5 in the next section).

The key element in the construction of the 8-attractor which enables us to conclude the monotonic behavior of orbits of the first return map, (and hence it enables the proof of Theorem 1 to work) is that the line segment $\overline{S_0S_1}$ is perpendicular to the imaginary axis. Our techniques presented in section 2 work for all cases of the relative position of S_0, and S_1 with respect to the imaginary axis provided that both S_0 and S_1 lie on the real. There are essentially four different cases (Figure 4) and in this section we briefly discuss the remaining three.

Theorem 2 (Case 2. Re $S_0 > 0 >$ Re S_1)**.** *Suppose that the centers of rotation S_0 and S_1 of the piecewise isometry T lie on the real axis and satisfy Re $S_0 > 0 >$ Re S_1. Then for every $x \in \mathbb{C}$, $|T^n x| \to \infty$.*

Theorem 3 (Case 3. $0 <$ Re $S_1 <$ Re S_0.)**.** *Suppose that the centers of rotations lie on the real axis and satisfy $0 <$ Re $S_1 <$ Re S_0. Then T restricted to $\overline{B(S_1, Re\ S_1)}$ is a rotation. However, for every $x \in \mathbb{C} - \overline{B(S_1, Re\ S_1)}$, $|T^n x| \to \infty$.*

Theorem 4 (Case 4. $0 <$ Re $S_0 <$ Re S_1.)**.** *Suppose that the centers of rotation lie on the real axis and satisfy $0 <$ Re $S_1 <$ Re S_0. Then the coding of every point under the piecewise isometry T is eventually constant - all orbits are eventually trapped in the right halfplane.*

The proofs of the above theorems use exactly the same ideas and calculations as in the part 1 of the proof of Theorem 1. We shall therefore outline only the main steps.

Case 2. One observes the orbit of every point is never trapped in one halfplane, hence the orbit has to alternate between atoms, and one can define the first return map $T_\triangle : P_0 \to P_0$ to the atom P_0. Then, as in step 2 in the proof of Theorem 1, one shows that $\mathrm{d}(T_\triangle x, S_0) \geq \mathrm{d}(x, S_0)$. Next, one shows that if $\mathrm{d}(T_\triangle^n x, S_0)$ is bounded (step 3, proof of Theorem 1), then x would have a periodic coding which turns out not to be possible (step 4, proof of Theorem 1). Hence, $|T^n x| \to \infty$.

Case 3. One observes that the orbit of every point is never trapped in P_0, hence the first return map $T_\triangle : P_1 \to P_1$ to the atom P_1 can be defined. Then, one shows that for $x \notin \overline{B(S_1, \mathrm{Re}\ S_1)}$, $\mathrm{d}(T_\triangle x, S_1) \geq \mathrm{d}(x, S_1)$. Following steps 3 and 4, one concludes that $|T^n x| \to \infty$.

Case 4. The orbit of a point cannot be trapped in the left halfplane. It must be trapped in the right halfplane, or the orbit has to alternate between atoms. In the latter case, one defines the first return map $T_\triangle : P_0 \to P_0$ to the atom P_0. Then, as in step 2, section 2, one shows that $\mathrm{d}(T_\triangle x, S_0) \le \mathrm{d}(x, S_0)$. Then as in the steps 3 and 4 in the proof of Theorem 1, one shows that the orbit must accumulate on the circles centered at S_0 and S_1 both tangent to the imaginary axis. This, however, unlike in case 1 (Theorem 1), is not possible as the circle centered at S_0 is contained in the atom P_1.

Finally, we note that Theorems 2 and 3 imply the following interesting result about the symbolic dynamics of our system.

Corollary 6. In cases 2 and 3, the codings of all points in $\mathbb{C}$ are irrational for Lebesgue almost all choices of parameters $(\alpha_0, \alpha_1) \in (0, 2\pi) \times (0, 2\pi)$.

The proof of this corollary immediately follows from the following proposition whose proof uses a short additional argument.

Proposition 3. *For Lebesgue almost all choices of $(\alpha_0, \alpha_1) \in (0, 2\pi)^2$, the coding of x is irrational under the piecewise isometry T if $|T^n x| \to \infty$ as $n \to \infty$.*

Proof. Suppose that α_0, α_1, π are linearly rationally independent and $|T^n x| \to \infty$ for some $x \in \mathbb{C}$. Suppose that the coding of x is eventually periodic, $\sigma(x) = w_0 \dots w_{j-1}(w_j \dots w_{j+k-1}) \dots$, where the block $(w_j \dots w_{j+k-1})$ is repeated infinitely many times. Then for all $m > 0$,

$$T^{j+m\,k} x = J^m(T^j x) \tag{9}$$

where $J = T_{w_{j+k-1}} T_{w_{j+k-2}} \cdot \ldots \cdot T_{w_j}$ is a composition of rotations by the angles α_0 and α_1. Since the isometry J preserves orientation and since α_0, α_1, π are linearly rationally independent, J cannot be a translation, hence it must be a rotation (by an angle noncommensurable with π). From (9) we conclude that the orbit $\{T^{j+m\,k} x : \ m > 0\}$ is bounded. Hence, $|T^n x| \not\to \infty$ as $n \to \infty$. □

6. Example of a system for which 8-attractor emerges as a local map.

In this section we illustrate an example of a piecewise rotation $T : \mathbb{C} \to C$ with partition $\{P_0, P_1\}$ into the left and right halfplane for which 8-attractor emerges as a local map. This example serves as a basis for the construction of more complicated piecewise isometric attractors.

Let $T_0, T_1 : \mathbb{C} \to \mathbb{C}$ be the rotations by angles $\alpha_0, \alpha_1 \in (0, 2\pi)$ about the centers: S_0 and S_1. Let $T : \mathbb{C} \to \mathbb{C}$ be the piecewise isometry with induced isometries $\{T_0, T_1\}$ and the partition $\{P_0, P_1\}$ of $\mathbb{C}$ into the left and right halfplane.

We set the parameters defining T so that there exists a periodic point S_2 lying on the real axis. Let $\alpha = \alpha_0 = \alpha_1$ be the common angle of rotation of T_0 and T_1 such that $-\pi/2 < \alpha < 0$ and α is not commensurable with π . Choose points S_0, S_2, S_1, S_2' to be the vertices of a rombus (Figure 5) such that: (i) S_0 and S_2 lie on the real axis, (ii) $|\langle S_2 S_0 S_2'| = |\langle S_2 S_1 S_2'| = |\alpha|$, (iii) $S_0, S_2' \in P_0$, $S_1, S_2 \in P_1$, (iv) $|\mathrm{Re}\ S_2'| > |\mathrm{Re}\ S_2|$.

Since $T(S_2) = T_1(S_2) = S_2'$ and $T(S_2') = T_0(S'_2) = S_2$, the point S_2 and the disc $D_2 = B(S_2, S_2)$ are periodic of period 2 and D_2 is tangent to the disc $D_1 = B(S_0, S_0)$. Let F be the intersection point of the positive imaginary axis and the circle $\partial B(S_2, |\mathrm{Re}\ S_2'|)$ (Figure 5). Define a neighborhood U of $D_0 \cup D_2$, $U = B(S_2, |\mathrm{Re}\ S_2'|) \cup B(S_0, |S_0 - F|)$. Let: $P_0' = U \cap P_0$ and $P_1' = U \cap P_1$ be a partition of U. Since $T P_0' \subset U$, and $T^2 P_1' \subset U$, we can define a local piecewise isometry $T_\triangle : U \to U$, $T_\triangle x = T_0 x$ if $x \in P_0'$ and $T_\triangle x = T_0 T_1 x$ if $x \in P_1'$. By Theorem 1 we conclude:

Proposition 4. *The local map $T_\triangle : U \to U$ is an 8-attractor map with partition: $\{P'_0, P_1'\}$, and induced isometries $T_0' = T_0 = T|_{P_0'}$ and $T_1' = T_0 T_1 = T|^2_{P_1'}$ (rotation about S_2 by 2α). Moreover, $\bigcap_{n\ge 0} T_\triangle^n U \cup Z = D_0 \cup D_2$ for some countable set Z and $D_0 \cup D_2$ is a piecewise isometric attractor for $T_\triangle$.*

Theorem 1 and Proposition 4 imply that

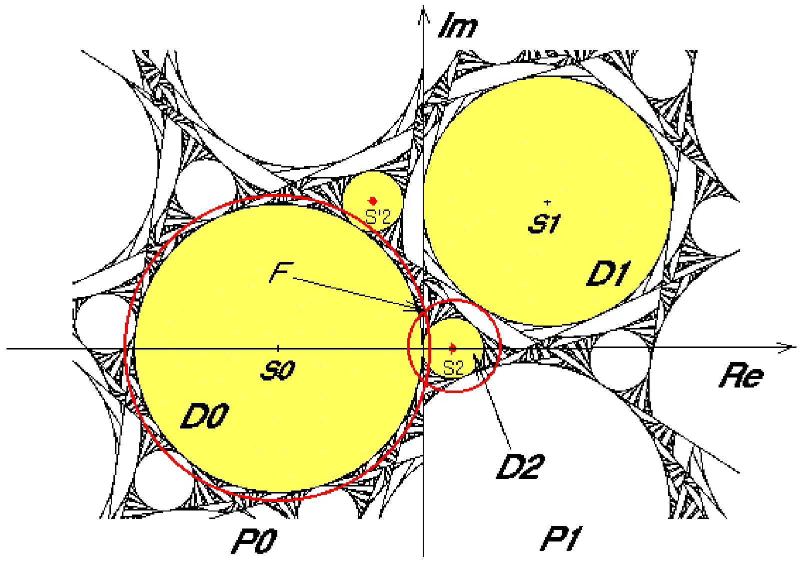

FIGURE 5. Cells and periodic discs in a piecewise rotation with two atoms whose right center of rotation lies away from the real axis. The small disc $D_2 = \overline{B(S_2, S_2)}$ is a periodic cell of period 2 and it is tangent to the fixed disc $D_1 = B(S_0, S_0)$. A local map $T_\triangle$ restricted to a neighborhood of $D_1 \cup D_2$ defines an 8-attractor. Also the "molecularly shaped" union of the discs: $M = D_0 \cup D_2 \cup T\,D_2$ is a piecewise isometric attractor for T. Points in the neighborhood of M whose orbits are not trapped in M accumulate on ∂M.

Corollary 7. The orbits of points in U under T are either trapped in one of the periodic discs or they accumulate on the union of the circles: $\partial D_0 \cup \partial D_2 \cup \partial T D_2$.

In conclusion, we note that Proposition 4 implies the existence of a new class of attractors consisting of more then two tangent discs.

Corollary 8. The molecularly shaped set $M = D_0 \cup D_2 \cup T\,D_2$ is a piecewise isometric attractor for T.

Proof. It is enough to show that for the open set $V = U \cup T_0^{-1} P_1' \supset D_0 \cup D_2 \cup T D_2$: (i) $T(V) \subset V$, (ii) $\bigcap_{n>0} T^n V \cup Z \subset M$ where Z is a countable set. The first assertion is clear. In order to show (ii), recall from part 2 in the proof of Theorem 1 that $\bigcap_{n>0} T^n V$ is the set of points in V initiating an infinite backward $T|_V$ chain. Similarly, $\bigcap_{n>0} T_\triangle^n U$ is the set of all points in U initiating an infinite backward $T_\triangle$ chain.

Suppose that $x_0 \in \bigcap_{n>0} T^n V$. The point x_0 initiates an infinite $T-$backward chain $\{x_{-k}\} \subset V$. If the forward T orbit of x_0 is trapped in P_0, then $x_0 \in D_0 - Z \subset M$. Suppose that the (forward T) orbit of x_0 alternates between P_0 and P_1 (the orbit of x_0 cannot be trapped in P_1). Let j be the smallest power such that $x_j = T^j x_0 \in P_1 \cap M$.

Let $\{x'_{-i}\} \subset \{x_{-j}\}$ be the sub-chain of all points x_{-j} such that either (a) $x_{-j} \in P_0 \cap M$ and $x_{-j-1} \in P_0 \cap M$, or (b) $x_{-j} \in P_0 \cap M$ and $x_{-j-l} \in P_1 \cap M$, or (c) $x_{-j} \in P_1 \cap M$ and $x_{-j-l} \in P_1 \cap M$, or (d) $x_{-j} \in P_1 \cap M$ and $x_{-j-1} \in P_0 \cap M$. Moreover, let $x'_0 = x_j$, and let $\{x'_{-i}\}$ preserve the natural order of $\{x_{-j}\}$. Then the sub-chain $\{x'_{-i}\}$ is infinite and contained in M and such that

$$T_\triangle x'_{-i} = x'_{-i+1},$$

which means that x_j initiates an infinite backward $T_\triangle$-chain in M. Hence, by Theorem 1 and Proposition 4, $\{x'_{-i}\} \subset \bigcap_{n>0} T_\triangle^n M = D_0 \cup D_2 - Z$. Finally, since x_0 is in the forward T orbit of $\{x'_{-i}\}$, x_0 is in the forward T orbit of $D_0 \cup D_2 - Z$, and thus $x_0 \in A - Z$. This shows (ii) and concludes the proof the corollary. □

Question. Using an additional argument we can show that all orbits of points in $\mathbb{C}$ must accumulate in $D_0 \cup D_1 \cup D_2 \cup TD_2$. Is it true that for all piecewise rotations with two atoms (with rotation angles α_0 and α_1 independent over rational multiples of π) all orbits must either escape to infinity, accumulate on the boundaries or inside the periodic discs?

7. Concluding Remarks and Open Questions

Other piecewise isometric attractors and classification of piecewise rotations with two atoms. In this article we described the dynamics of a piecewise isometric attractor consisting of two discs. Theorem 1 and the example in section 5, illustrating that 8-attractor emerges as a local model in piecewise isometries and gives rise to more complicated ("molecular") attractors consisting of many tangent discs, can serve as a basis for the study of other more complicated attractors. In a separate article, we plan to investigate these molecular attractors and show that they appear for a dense set of small perturbation of the 8-attractor map. We hope that the study of the mechanisms of how the attractor emerges would give a deeper insight and lead to the classification of all piecewise rotations with two atoms. Moreover, we hope that this study will give insight into the dynamics of the piecewise rotation on the torus: $T : [0,1] \times [0,1] \to [0,1] \times [0,1]$, $Tx = Mx \bmod 1, i$ where $M \in SO(\mathbb{R}, 2)$ (the entries in M are not integers, hence this is not a toral automorphism) where molecular attractors have been experimentally observed on the computer.

Symbolic word growth. One of the very fascinating subjects in the studies of the dynamics of piecewise isometric systems is the symbolic word growth of all finite words realizable by the piecewise isometry T. Symbolic growth measures the complexity of the system and has recently been very vigorously studied by a number of researchers particularly interested in the "slow" growth. Let $f(n)$ denote be the number of words of length n in the set W_T (W_T is the set of all finite words realizable by T).

Question. What is the growth of $f(n)$ for noninvertible piecewise isometries?

This growth of $f(n)$ is related to the topological entropy of $\hat{T} : \hat{X} \to \hat{X}$. If $\hat{T}$ has zero entropy, then $f(n)$ is a subexponential function (the shift map restricted to $\overline{\sigma(X)}$ is a factor map for $\hat{T}$ (Corollary 1). We conjecture that:

Conjecture 1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise isometry with convex partition $\mathcal{P}$. Then $\hat{T}$ has zero topological entropy. Hence T has a subexponential symbolic growth complexity.*

Using ideas in [11] and the following a version of Corollary 4 and the variantional principle we are able to affirm that $h(\hat{T}) = 0$ in the case of the 8-attractor map.

Proposition 5. *Every non-atomic probability Borel invariant with respect to $\hat{T}$ measure μ on $\hat{Y}$ is supported on the set $S = \{(x, \sigma(x)) : x \in Y, \sigma(x)$ is constant$\}$.*

The proof of Proposition 5 uses Corollary 4 and the argument very similar to step 3 in section 2 claiming that points in $\hat{Y} \subset \overline{(X, \Omega)}$ whose orbits under $\hat{T}$ project onto the discontinuity axis infinitely often must follow the eventually periodic coding pattern. Hence, there are at most a countable number of points in $\hat{Y} \subset \overline{(X, \Omega)}$ whose iterates project onto the discontinuity axis infinitely often [7]. From the measure-theoretic viewpoint non-atomic invariant measures for $\hat{T} : \hat{Y} \to \hat{Y}$ can be identified with non-atomic invariant measures for $T : Y \to Y$.

Recently, Gutkin and Haydn [8] established that for a subclass of invertible two-dimensional piecewise isometries the entropy is zero. Their setting, however, does not include orbits which intersecting the boundary of the atoms.

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