AN APPROXIMATE HERBRAND'S THEOREM AND DEFINABLE FUNCTIONS IN METRIC STRUCTURES

ISAAC GOLDBRING

Abstract. We develop a version of Herbrand's theorem for continuous logic and use it to prove that definable functions in infinite-dimensional Hilbert spaces are piecewise approximable by affine functions. We obtain similar results for definable functions in Hilbert spaces expanded by a group of generic unitary operators and Hilbert spaces expanded by a generic subspace. We also show how Herbrand's theorem can be used to characterize definable functions in absolutely ubiquitous structures from classical logic.

1. Introduction

The main motivation for this paper comes from the study of definable functions in metric structures; this study was initiated by the author in [11], where a study of the definable functions in Urysohn's metric space was undertaken, and continued in [10], where the definable linear operators in (infinite-dimensional) Hilbert spaces were characterized. However, lacking any understanding of arbitrary definable functions in Hilbert spaces, we conjectured that they were, in some sense, "piecewise affine" in analogy with the classical case of an infinite vector space over a division ring. In unpublished lecture notes by van den Dries on motivic integration [9], we came upon a proof of the piecewise affineness of definable functions in such vector spaces using the following classical theorem of Herbrand:

Theorem 1.1 (Herbrand [12]). *Suppose that $\mathcal{L}$ is a first-order signature and T is a universal $\mathcal{L}$-theory with quantifier elimination. Let $\varphi(\vec{x},\vec{y})$ be a formula, where $\vec{x}=(x_1,\ldots,x_m)$, $y=(y_1,\ldots,y_n)$, $m\geq 1$. Then there are $\mathcal{L}$-terms*

$$t_{11}(\vec{x}),\ldots,t_{1n}(\vec{x}),\ldots,t_{k1}(\vec{x}),\ldots,t_{kn}(\vec{x}),\quad (k\in\mathbb{N}^{>0})$$

such that

$$T\models\forall\vec{x}\forall\vec{y}\left(\varphi(\vec{x},\vec{y})\rightarrow\bigvee_{i=1}^{k}\varphi(\vec{x},t_{i1}(\vec{x}),\ldots,t_{in}(\vec{x}))\right).$$

Although this theorem is not immediately applicable to the case of an infinite vector space V over a division ring (for the axioms expressing that V is infinite are existential), Herbrand's theorem does apply to the theory

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of V with constants added for names of elements of V. Since terms in this extended language name affine functions, we get the aforementioned characterization of definable functions in V. (According to van den Dries, this use of Herbrand's theorem is well-known and often used.) Although Theorem 1.1 has an easy model-theoretic proof using compactness, we should remark that the result was first established using proof-theoretic techniques; see [6] and [7] for more on the history of Herbrand's result.

In this paper, we prove a version of Herbrand's theorem for continuous logic (Theorem 2.7 and Corollary 2.8 below) and use it to characterize definable functions in Hilbert spaces and some of their generic expansions, proving, in the case of pure Hilbert spaces, that definable functions are "piecewise approximable by affine functions." Along the way, we note that this method actually works whenever T is a model-complete $\exists\forall$-axiomatizable theory. In particular, we show that one can use Herbrand's theorem to understand definable functions in absolutely ubiquitous structures from classical logic.

Ulrich Kohlenbach pointed out to me that there is a proof-theoretic approach to metric structures (including Hilbert spaces) which deals with issues (e.g. Gödel's functional interpretation) which can be viewed as far reaching generalizations of Herbrand's theorem; see [13] and [14]. It would be interesting to understand the connection between these approaches.

We assume that the reader is familiar with the basic definitions of continuous logic as presented in the survey article [2]. In particular, each predicate take values in a closed, bounded interval in $\mathbb{R}$.

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2. Herbrand's Theorem in Continuous Logic

In this section, we let $\mathcal{L}$ denote an arbitrary continuous signature. Recall that an $\mathcal{L}$-theory is a set of closed $\mathcal{L}$-conditions. Given an $\mathcal{L}$-theory T and an $\mathcal{L}$-structure $\mathcal{M}$, we write $\mathcal{M} \models T$ to mean $\sigma^{\mathcal{M}} = 0$ whenever the condition "$\sigma = 0$" is in T. We extend this notation to sets of sentences in the obvious way: given a set of $\mathcal{L}$-sentences Γ and an $\mathcal{L}$-structure $\mathcal{M}$, we write $\mathcal{M} \models \Gamma$ to mean $\sigma^{\mathcal{M}} = 0$ for all $\sigma \in \Gamma$. It then makes sense to say that the collection of sentences Γ axiomatizes the theory T: for all $\mathcal{L}$-structures $\mathcal{M}$, $\mathcal{M} \models \Gamma$ if and only if $\mathcal{M} \models T$.

Definition 2.1. Suppose that Δ is a set of $\mathcal{L}$-sentences.

(1) We say that Δ is *closed under min* if whenever $\sigma_1, \ldots, \sigma_n$ are sentences from Δ, we have $\min_{1 \leq i \leq n} \sigma_i \in \Delta$.
(2) We say that Δ is *closed under weakening* if whenever $\sigma \in \Delta$, then $\sigma \dot{-} r \in \Delta$ for every $r \in [0, 1]$.

The following lemma is in a similar spirit to Lemma 3.4 of [18]; the classical version, whose proof we mimic, can be found in [8].

Lemma 2.2. *Suppose that T is a satisfiable $\mathcal{L}$-theory and Δ is a set of $\mathcal{L}$-sentences that is closed under min and weakening. Then the following are equivalent:*

(1) *T is axiomatizable by a collection of sentences $\Gamma \subseteq \Delta$;*
(2) *For all $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ satisfying $\mathcal{M} \models T$ and $\sigma^{\mathcal{N}} = 0$ for all $\sigma \in \Delta$ with $\sigma^{\mathcal{M}} = 0$, we have $\mathcal{N} \models T$.*

Proof. Clearly (1) $\Rightarrow$ (2), so we need to prove (2) $\Rightarrow$ (1). Consider the set $\Gamma = \{\sigma \ : \ \sigma \in \Delta \text{ and } T \models \sigma\}$. We claim that Γ axiomatizes T. Suppose $\mathcal{N} \models \Gamma$. Let

$$\Sigma = \{\frac{r}{2} \dot{-} \delta \ : \ \delta^{\mathcal{N}} = r, \ r > 0, \ \delta \in \Delta\}.$$

We claim that $T \cup \Sigma$ is consistent. Suppose otherwise. Then there are $\delta_1, \ldots, \delta_k, \ r_1, \ldots, r_k$ such that $T \models \min_{1 \leq i \leq k}(\delta_i \dot{-} \frac{r_i}{2}) = 0$. Since Δ is closed under min and weakening, we have that $\min_{1 \leq i \leq k}(\delta_i \dot{-} \frac{r_i}{2}) \in \Gamma$, so $\mathcal{N} \models \min_{1 \leq i \leq k}(\delta_i \dot{-} \frac{r_i}{2}) = 0$, which is a contradiction to the fact that $\delta_i^{\mathcal{N}} = r_i$ for each i. Let $\mathcal{M} \models T \cup \Sigma$. Now suppose that $\sigma \in \Delta$ and $\sigma^{\mathcal{M}} = 0$. Then $\sigma^{\mathcal{N}} = 0$, else $\frac{r}{2} \dot{-} \sigma \in \Sigma$ for some $r > 0$, contradicting $\sigma^{\mathcal{M}} = 0$. By (2), we have $\mathcal{N} \models T$. $\square$

Given an $\mathcal{L}$-structure $\mathcal{M}$, let $\mathrm{D}(\mathcal{M})$ be the set of closed $\mathcal{L}(\mathcal{M})$-conditions of the form $\sigma = 0$, where σ is a quantifer-free $\mathcal{L}(\mathcal{M})$ sentence and $\sigma^{\mathcal{M}} = 0$; this is just the *quantifier-free diagram of* $\mathcal{M}$. The following lemma is proved just as in classical logic.

Lemma 2.3. *If $\mathcal{N} \models D(\mathcal{M})$, then the $\mathcal{L}$-reduct of $\mathcal{N}$ contains a substructure isomorphic to $\mathcal{M}$.*

Let us call a sentence σ *universal* if it is *formally universal*, that is, of the form $\sup_{\vec{x}} \varphi(\vec{x})$, where φ is quantifier-free, or else logically equivalent to a formally universal sentence.

Lemma 2.4. *The set of universal sentences is closed under min and weakening.*

Proof. It is readily checked that $\min(\sup_{\vec{x}} \varphi(\vec{x}), \sup_{\vec{y}} \psi(\vec{y}))$ is logically equivalent to $\sup_{\vec{x},\vec{y}} \min(\varphi(\vec{x}), \psi(\vec{y}))$ and $(\sup_{\vec{x}} \varphi(\vec{x})) \dot{-} r$ is logically equivalent to $\sup_{\vec{x}}(\varphi(\vec{x}) \dot{-} r)$. $\square$

If Γ is a set of closed $\mathcal{L}$-conditions, we set

$$\Gamma^+ := \{``\sigma \leq \frac{1}{n}" \ : \ \sigma \in \Gamma, n \geq 1\}.$$

Corollary 2.5. *Given an $\mathcal{L}$-theory T, the following are equivalent:*

(1) *T has a universal axiomatization;*
(2) *For any $\mathcal{M} \models T$ and substructure $\mathcal{N}$ of $\mathcal{M}$, we have $\mathcal{N} \models T$.*

Proof. Clearly (1) implies (2), so we prove that (2) implies (1). We use the criterion developed in Lemma 2.2 applied to the set of universal sentences. Suppose that $\mathcal{M} \models T$ and for all universal sentences σ, we have $\sigma^{\mathcal{M}} = 0$ implies $\sigma^{\mathcal{N}} = 0$. We want $\mathcal{N} \models T$. Let $T' = T \cup \mathrm{D}(\mathcal{N})^+$. We claim that T' is satisfiable. Fix atomic $\mathcal{L}(\mathcal{N})$-sentences $\sigma_1(\vec{b}), \ldots, \sigma_n(\vec{b})$ such that $\sigma_i^{\mathcal{N}}(\vec{b}) = 0$. Then $\mathcal{N} \models \inf_{\vec{x}} \max(\sigma_i(\vec{x})) = 0$. Suppose, towards a contradiction, that $\mathcal{M} \not\models \inf_{\vec{x}} \max(\sigma_i(\vec{x})) = 0$. Then there is $r \in (0,1]$ such that $\mathcal{M} \models \sup_{\vec{x}}(r \dot{-} \max(\sigma_i(\vec{x}))) = 0$. By assumption, we have $\mathcal{N} \models \sup_{\vec{x}}(r \dot{-} \max(\sigma_i(\vec{x}))) = 0$, which is a contradiction. Consequently, for any $k \geq 1$, there is $\vec{a} \in M$ such that $\mathcal{M} \models \max(\sigma_i(\vec{a})) \leq \frac{1}{k}$. It follows by compactness that T' is satisfiable. Let $\mathcal{A}' \models T'$ and let $\mathcal{A}$ be the $\mathcal{L}$-reduct of $\mathcal{A}'$. Then $\mathcal{A} \models T$ and $\mathcal{N}$ is (isomorphic to) a substructure of $\mathcal{A}$, whence $\mathcal{N} \models T$. □

Definition 2.6. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\langle A \rangle_0$ be the $\mathcal{L}$-prestructure generated by A. Then the closure of $\langle A \rangle_0$ in M is the completion of $\langle A \rangle_0$, whence a substructure of $\mathcal{M}$, called the *substructure of $\mathcal{M}$ generated by A.*

By Theorem 3.5 of [2], any $\mathcal{L}$-formula $\varphi(\vec{x})$ has a modulus of uniform continuity $\Delta_\varphi : (0,1] \to (0,1]$, that is, for any $\mathcal{L}$-structure $\mathcal{M}$, any $\epsilon > 0$, and any tuples $\vec{a}, \vec{b}$ from M, if $d(\vec{a}, \vec{b}) < \Delta_\varphi(\epsilon)$, then $|\varphi^{\mathcal{M}}(\vec{a}) - \varphi^{\mathcal{M}}(\vec{b})| \leq \epsilon$.

Theorem 2.7 (Continuous Herbrand Theorem)**.** *Suppose that T is a universal $\mathcal{L}$-theory that admits quantifier-elimination. Let $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$. Then for any formula $\varphi(\vec{x}, \vec{y})$ and any $\epsilon > 0$, there are $\mathcal{L}$-terms*

$$t_{11}(\vec{x}), \ldots, t_{1n}(\vec{x}), \ldots, t_{k1}(\vec{x}), \ldots, t_{kn}(\vec{x}) \quad (k \in \mathbb{N}^{>0})$$

such that, for any $\mathcal{M} \models T$ and any $\vec{a} \in M^m$, if $\mathcal{M} \models \inf_{\vec{y}} \varphi(\vec{a}, \vec{y}) = 0$, then

$$\mathcal{M} \models \min_{1 \leq i \leq k} \varphi(\vec{a}, t_{i1}(\vec{a}), \ldots, t_{in}(\vec{a})) \leq \epsilon.$$

Proof. Consider the set of closed $\mathcal{L}$-conditions $\Gamma(\vec{x})$ given by

$$T \cup \{\inf_{\vec{y}} \varphi(\vec{x}, \vec{y}) = 0\} \cup \{\varphi(\vec{x}, t_1(\vec{x}), \ldots, t_n(\vec{x})) \geq 2\epsilon \, : \, t_1(\vec{x}), \ldots, t_n(\vec{x}) \ \mathcal{L}\text{-terms}\}.$$

By compactness, it is enough to prove that Γ is unsatisfiable. Suppose, towards a contradiction, that $\mathcal{M} \models \Gamma(\vec{a})$, where $\vec{a} = (a_1, \ldots, a_m) \in M^m$. Let $\mathcal{N}$ be the substructure of $\mathcal{M}$ generated by $\{a_1, \ldots, a_m\}$. Then $\mathcal{N} \models T$, whence $\mathcal{N} \preceq \mathcal{M}$ by model-completeness. Consequently, $\mathcal{N} \models \inf_{\vec{y}} \varphi(\vec{a}, \vec{y})$. Fix $\delta \leq \frac{\epsilon}{2}$. Take $\vec{b} \in N^n$ such that $\varphi^{\mathcal{N}}(\vec{a}, \vec{b}) \leq \delta$. For each i, let $t_i(\vec{x})$ be a term so that $d(t_i(\vec{a}), \vec{b}_i) < \Delta_\varphi(\delta)$, whence $\varphi^{\mathcal{N}}(\vec{a}, t_1(\vec{a}), \ldots, t_n(\vec{a})) \leq 2\delta \leq \epsilon$. By model-completeness again, $\varphi^{\mathcal{M}}(\vec{a}, t_1(\vec{a}), \ldots, t_m(\vec{a})) \leq \epsilon$, which is a contradiction to the fact that $\mathcal{M} \models \Gamma(\vec{a})$. □

The following rephrasing of the previous theorem more closely resembles the usual statement of Herbrand's theorem.

Corollary 2.8. *Suppose that T is a universal $\mathcal{L}$-theory that admits quantifier-elimination. Let $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$. Then for any formula $\varphi(\vec{x}, \vec{y})$ and any $\epsilon > 0$, there are $\mathcal{L}$-terms*

$$t_{11}(\vec{x}), \ldots, t_{1n}(\vec{x}), \ldots, t_{k1}(\vec{x}), \ldots, t_{kn}(\vec{x}) \quad (k \in \mathbb{N}^{>0})$$

and an increasing continuous function $\alpha : [0,1] \to [0,1]$ satisfying $\alpha(0) = 0$ such that

$$T \models \sup_{\vec{x}}((\min_{1 \le i \le k} \varphi(\vec{x}, t_{i1}(\vec{x}), \ldots, t_{in}(\vec{x})) \dot{-} \epsilon) \dot{-} \alpha(\inf_{\vec{y}}(\varphi(\vec{x}, \vec{y}))) = 0.$$

Proof. This is immediate from the preceding theorem and Proposition 7.15 of [2]. □

3. Primitive theories

In this short section, $\mathcal{L}$ continues to denote an arbitrary (continuous) signature and T denotes an $\mathcal{L}$-theory.

Definition 3.1. Following [15] (in the classical setting), we say that T is *primitive* if there exists sets of closed $\mathcal{L}$-conditions Γ and Δ, where Γ consists of universal conditions and Δ consists of existential conditions, such that $\Gamma \cup \Delta$ axiomatizes T.

Remark 3.2. In classical logic, it is mentioned in [15] that T is primitive if and only if: whenever $\mathcal{M}_0, \mathcal{M}_1 \models T$ and $\mathcal{M}_0 \subseteq \mathcal{N} \subseteq \mathcal{M}_1$, then $\mathcal{N} \models T$. It is also mentioned in [15] that T is $\exists\forall$-axiomatizable if and only if: whenever $\mathcal{M}_0, \mathcal{M}_1 \models T$, $\mathcal{M}_0 \preceq \mathcal{M}_1$, and $\mathcal{M}_0 \subseteq \mathcal{N} \subseteq \mathcal{M}_1$, then $\mathcal{N} \models T$. It follows that for model-complete theories T, T is primitive if and only if T is $\exists\forall$-axiomatizable. An interesting example of a model-complete $\exists\forall$-theory is Example 3 of [16].

Proposition 3.3. *Suppose that T is a complete, model-complete primitive $\mathcal{L}$-theory. Let $\mathcal{M} \models T$ and let $T_{\mathcal{M}}$ be the $\mathcal{L}(\mathcal{M})$-theory of $\mathcal{M}$. Then $T_{\mathcal{M}}$ is universally axiomatizable and admits quantifier-elimination.*

Proof. Let Γ be a set of universal sentences and Δ a set of existential sentences such that $\Gamma \cup \Delta$ axiomatizes T. In order to prove that $T_{\mathcal{M}}$ has a universal axiomatization, it suffices to prove that $T_{\mathcal{M}}$ is axiomatized by $\Gamma \cup \mathrm{D}(\mathcal{M})$. Suppose that $\mathcal{N} \models \Gamma \cup \mathrm{D}(\mathcal{M})$. Then $\mathcal{M}$ is a substructure of $\mathcal{N}$. Now any axiom from Δ is true in $\mathcal{N}$ since it is witnessed by things in $\mathcal{M}$. Consequently, $\mathcal{N} \models T$, whence $\mathcal{N} \models T_{\mathcal{M}}$ by model-completeness of T. Clearly, $T_{\mathcal{M}}$ is still model-complete; since model-completeness and quantifier elimination are equivalent for universal theories, we have that $T_{\mathcal{M}}$ admits quantifier elimination. □

The following proposition explains how we use Herbrand's theorem in connection with definable functions.

Proposition 3.4. *Suppose that T is a complete, model-complete primitive $\mathcal{L}$-theory. Suppose $\mathcal{M} \models T$ and $f : M^n \to M$ is a definable function. Then for any $\epsilon > 0$, there are $\mathcal{L}(M)$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ such that: for all $\vec{a} \in M^n$, there is $i \in \{1, \ldots, k\}$ with $d(f(\vec{a}), t_i(\vec{a})) \leq \epsilon$.*

Proof. Fix $\epsilon > 0$. Let $\varphi(\vec{x}, y)$ be an $\mathcal{L}(M)$-formula such that

$$|d(f(\vec{a}), b) - \varphi^{\mathcal{M}}(\vec{a}, b)| \leq \frac{\epsilon}{3}$$

for all $\vec{a} \in M^n$ and $b \in M$. By Herbrand's theorem applied to $T_{\mathcal{M}}$ (which is applicable by Proposition 3.3), there are $\mathcal{L}(M)$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ such that, for all $\vec{a} \in M^n$, if $\mathcal{M} \models \inf_y(\varphi(\vec{a}, y) \dot{-} \frac{\epsilon}{3}) = 0$, then

$$\mathcal{M} \models (\varphi(\vec{a}, t_i(\vec{a})) \dot{-} \frac{\epsilon}{3}) \leq \frac{\epsilon}{3}$$

for some $i \in \{1, \ldots, k\}$. Notice that the antecedent of the preceding conditional statement holds since $\varphi^{\mathcal{M}}(\vec{a}, f(\vec{a})) \leq \frac{\epsilon}{3}$. Consequently, for every $\vec{a} \in M^n$, there is $i \in \{1, \ldots, k\}$ such that $d(f(\vec{a}), t_i(\vec{a})) \leq \epsilon$. □

Remark 3.5. Fix a definable function $f : M^n \to M$. Fix $\epsilon > 0$ and let the $\mathcal{L}(\mathcal{M})$-terms $t_1(\vec{x}), \ldots, t_k(\vec{x})$ be as in the conclusion of the previous proposition. Suppose that $\mathcal{M} \preceq \mathcal{N}$ and $f : N^n \to N$ is the natural extension of f to a definable function in $\mathcal{N}$. Then, for every $\vec{a} \in N^n$, there is $i \in \{1, \ldots, k\}$ such that $d(f(\vec{a}), t_i(\vec{a})) \leq \epsilon$. Indeed, repeat the proof of the preceding proposition, using Corollary 2.8 instead of Theorem 2.7.

We end this section with an application to classical logic. A source of primitive theories in classical logic comes from the notion of an *absolutely ubiquitous* structure. Suppose that $\mathcal{L}$ is a finite first-order signature and $\mathcal{M}$ is a countable $\mathcal{L}$-structure. Recall that $\mathcal{M}$ is said to be *locally finite* if every finitely generated substructure of $\mathcal{M}$ is finite and $\mathcal{M}$ is said to be *uniformly locally finite* if there is a function $g : \mathbb{N}^{>0} \to \mathbb{N}^{>0}$ such that, for all $A \subseteq M$, if $|A| \leq n$, then $|\langle A \rangle| \leq g(n)$, where $\langle A \rangle$ denotes the substructure of $\mathcal{M}$ generated by A. Also recall that the *age of* $\mathcal{M}$, denoted $\mathrm{Age}(\mathcal{M})$, is the set of isomorphism classes of finitely generated substructures of $\mathcal{M}$. Finally, we say that $\mathcal{M}$ is absolutely ubiquitous if:

(1) $\mathcal{M}$ is uniformly locally finite, and
(2) whenever $\mathcal{N}$ is a countable, locally finite $\mathcal{L}$-structure with $\mathrm{Age}(\mathcal{M}) = \mathrm{Age}(\mathcal{N})$, then $\mathcal{M} \cong \mathcal{N}$.

It follows immediately from the definition that if $\mathcal{M}$ is an absolutely ubiquitous $\mathcal{L}$-structure and $T := \mathrm{Th}(\mathcal{M})$, then T is primitive and $\aleph_0$-categorical, whence model-complete (see also Lemma 2.1 of [17]). Consequently, T meets the hypotheses of Proposition 3.4. It follows that *definable functions in absolutely ubiquitous structures are piecewise given by terms.* In particular, if G is an absolutely ubiquitous (pure) group (these are discussed at length in [17]), then definable functions are piecewise given by words, that is, if $f : G^n \to G$

is a definable function, then there are words $w_1(x,y),\ldots,w_k(x,y)$, and parameters b from G such that, for each $a \in G^n$, $f(a) = w_i(a,b)$ for some $i \in \{1,\ldots,k\}$.

4. Applications

In this section, we present some continuous, model-complete primitive theories (which actually have quantifier-elimination) and use Proposition 3.4 above to understand the definable functions in models of these theories.

Until further notice, we suppose that $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and we set

$$\mathbb{D} := \{\lambda \in \mathbb{K} \ : \ |\lambda| \leq 1\}.$$

Also, $\mathcal{L}$ denotes the (1-sorted) continuous signature for unit balls of $\mathbb{K}$-Hilbert spaces. More specifically, when $\mathbb{D} = \mathbb{R}$, $\mathcal{L}$ contains:

- a constant symbol 0;
- a binary function symbol $f_{\alpha,\beta}$ for every $\alpha,\beta \in \mathbb{D}$ with $|\alpha| + |\beta| \leq 1$;
- a binary predicate symbol $\langle \cdot,\cdot \rangle$ that takes values in $[-1,1]$.

If $\mathbb{D} = \mathbb{C}$, then rather than having one predicate symbol for the inner product, we have two: one for the real part and one for the imaginary part.

If H is a $\mathbb{K}$-Hilbert space, the unit ball of H, $B_1(H)$, is naturally an $\mathcal{L}$-structure, where 0 is interpreted as the zero vector of H, $f_{\alpha,\beta}$ is interpreted as the function $(x,y) \mapsto \alpha x + \beta y$, and $\langle \cdot,\cdot \rangle$ is interpreted as the inner product of H. For sake of readability, we often write H instead of $B_1(H)$ when speaking of this way of treating $B_1(H)$ as an $\mathcal{L}$-structure.

Let T be the $\mathcal{L}$-theory of (the unit ball of) an infinite-dimensional $\mathbb{K}$-Hilbert space. Then T is primitive as the Hilbert space axioms are universal and the axioms for infinite-dimensionality are existential. We should remark that we could work in the many-sorted setting for Hilbert spaces (as in [10]) if we name the bijections $x \mapsto nx : B_1 \to B_n$ and $x \mapsto \frac{1}{n}x : B_n \to B_1$, for then the resulting theory of Hilbert spaces is universal.

In the rest of this subsection, $H \models T$ and H^* is an elementary extension of H. In order to make any sense of Proposition 3.4 in this context, we must first understand $\mathcal{L}(H)$-terms.

Lemma 4.1. *If $t(x)$ is an $\mathcal{L}(H)$-term, then there are $\lambda \in \mathbb{D}$ and $v \in B_1(H)$ so that $t(a) = \lambda a + v$ for all $a \in B_1(H)$.*

Proof. One proves this by induction on the complexity of $t(x)$, the base case being immediate. Now suppose that $t_i(x) = \lambda_i x + v_i$ for $i = 1,2$ and α,β are so that $|\alpha| + |\beta| \leq 1$. Then

$$f_{\alpha,\beta}(t_1(a),t_2(a)) = \alpha t_1(a) + \beta t_2(a) = (\alpha\lambda_1 + \beta\lambda_2)a + (\alpha v_1 + \beta v_2).$$

It remains to observe that $|\alpha\lambda_1 + \beta\lambda_2| \leq 1$. $\square$

Corollary 4.2. *Let $f : H \to H$ be definable. Then given $\epsilon > 0$, there are $\lambda_1,\ldots,\lambda_k \in \mathbb{D}$ and $v_1,\ldots,v_k \in B_1(H)$ such that, for all $a \in B_1(H^*)$, there is $i \in \{1,\ldots,k\}$ with $d(f(a),\lambda_i a + v_i) \leq \epsilon$.*

Fix $a \in B_1(H^*)$. Then there are sequences (λ_n) from $\mathbb{D}$ and (v_n) from $B_1(H)$ with $\lambda_n a + v_n \to f(a)$ as $n \to \infty$. By taking subsequences, we may suppose that $\lambda_n \to \lambda \in \mathbb{D}$. It then follows that (v_n) is a Cauchy sequence in $B_1(H)$, whence $v_n \to v \in B_1(H)$. It follows that $f(a) = \lambda a + v$. We have just proven the following result:

Corollary 4.3. *For any $a \in B_1(H^*)$, there are $\lambda \in \mathbb{D}$ and $v \in B_1(H)$ such that $f(a) = \lambda a + v$.*

Corollary 4.4. *Suppose that H^* is ω_1-saturated and $f(H^\perp) \subseteq H^\perp$. Fix $\epsilon > 0$ and let $\lambda_1, \ldots, \lambda_m$ be a finite ϵ-net for $\mathbb{D}$. Then there is a finite-dimensional subspace K of H such that, for all $a \in B_1(H^*) \cap K^\perp$, there is $i \in \{1, \ldots, m\}$ such that $d(f(a), \lambda_i a) < \epsilon$.*

Proof. Let $a \in B_1(H^*) \cap H^\perp$. Take $\lambda \in \mathbb{D}$ and $v \in B_1(H)$ such that $f(a) = \lambda a + v$. Then

$$0 = \langle f(a), v\rangle = \langle \lambda a + v, v\rangle = \langle v, v\rangle.$$

Thus, $f(a) = \lambda a$. Let (a_n) be an orthonormal basis for H. Then the following set of conditions is unsatisfiable in H^*:

$$\{\langle x, a_n\rangle = 0 \ : \ n < \omega\} \cup \{d(f(x), \lambda_i x) \geq \epsilon \ : \ i = 1, \ldots, m\}.$$

By saturation, there is $n < \omega$ such that, setting $K := \operatorname{span}(a_1, \ldots, a_n)$, we have $d(f(x), \lambda_i x) < \epsilon$ for all $x \in B_1(H^*) \cap K^\perp$. □

How does Corollary 4.2 relate to functions definable in the many-sorted language for Hilbert spaces considered in [10]? In order to elucidate this, we first clarify how the syntax of continuous logic works in the case that the predicates take values in intervals other than $[0, 1]$. (This is omitted in the survey [2] and was communicated to me by Ward Henson.) Let $\mathcal{L}'$ be a many-sorted (continuous) signature with sort set S. In particular, one associates to each predicate symbol P of $\mathcal{L}$ a closed, bounded interval I_P in $\mathbb{R}$. Then one also associates to each formula φ a closed, bounded interval I_φ in $\mathbb{R}$ as follows:

- Given two terms $t_1(\vec{x})$ and $t_2(\vec{x})$ of arity $(s_1, \ldots, s_n, s_{n+1})$, the formula $\varphi(\vec{x}) = d(t_1(\vec{x}), t_2(\vec{x}))$ is an atomic formula with $I_\varphi := [0, N]$, where N is the bound on the metric of sort s_{n+1}.
- If P is a predicate symbol of arity $(s_1, \ldots, s_n)$ and $t_1(\vec{x}), \ldots, t_n(\vec{x})$ are terms such that t_i takes values in sort s_i, then the formula $\varphi(\vec{x}) = P(t_1(\vec{x}), \ldots, t_n(\vec{x}))$ is an atomic formula with $I_\varphi := I_P$.
- Suppose that $\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x})$ are formulae with associated intervals $I_{\varphi_1}, \ldots, I_{\varphi_n}$. Suppose that u is a continuous function with domain $I_{\varphi_1} \times \cdots \times I_{\varphi_n}$ and range I, a closed, bounded interval in $\mathbb{R}$. Then $\varphi(\vec{x}) = u(\varphi_1(\vec{x}), \ldots, \varphi_n(\vec{x}))$ is a formula with $I_\varphi := I$.
- If φ is a formula with associated interval I_φ, then $\psi = \sup_x \varphi$ is a formula with $I_\psi := I_\varphi$. Similarly for $\inf_x \varphi$.

For an interval $I = [a,b] \subseteq \mathbb{R}$ with $a < b$, define $u_I : I \to [0,1]$ by $u_I(x) := \frac{1}{b-a}(x-a)$. Note that u_I is a homeomorphism with inverse $u_I^{-1}(x) = a+(b-a)x$.

We let $\mathcal{L}_{\mathrm{ms}}$ denotes the many-sorted theory of Hilbert spaces used in [10].

Lemma 4.5. *For any quantifier-free $\mathcal{L}_{\mathrm{ms}}$-formula $\varphi(\vec{x})$, where $\vec{x}$ is a tuple of variables of sort $B_1(H)$, there is a quantifier-free $\mathcal{L}$-formula $\psi(\vec{x})$ with $I_\psi = [0,1]$ such that*

$$H \models \sup_{\vec{x}} |u_{I_\varphi}(\varphi(\vec{x})) - \psi(\vec{x})| = 0.$$

In particular, when $I_\varphi = [0,1]$, we have $H \models \sup_x |\varphi(\vec{x}) - \psi(\vec{x})| = 0$.

Proof. The proof goes by induction on the complexity of φ, the main work taking place in the case when φ is atomic, which involves a painful case distinction. Let us illustrate the idea by considering terms $t_i(x,y) = \lambda_i x + \mu_i y$ $(i = 1,2)$ where $|\lambda_i|, |\mu_i| \le n$. (In the general situation, terms can be much more complicated due to the number of variables and the inclusion maps.) First suppose that $\varphi(x,y) = d(t_1(x,y), t_2(x,y))$. Since each t_i takes values in B_{2n}, we have $I_\varphi = [0,4n]$. Then $I_\varphi(\varphi(x,y)) = \frac{1}{4n} d(t_1(x,y), t_2(x,y))$. Let $\psi(x,y) = \|\frac{\lambda_1-\lambda_2}{4n}x + \frac{\mu_1-\mu_2}{4n}y\|$. Since $|\frac{\lambda_1-\lambda_2}{4n}| + |\frac{\mu_1-\mu_2}{4n}| \le 1$, we have that ψ is an $\mathcal{L}$-formula with $I_\psi = [0,1]$. Clearly ψ is as desired.

Now suppose that $\varphi(x,y) = \langle t_1(x,y), t_2(x,y)\rangle$. Now $I_\varphi = [-4n^2, 4n^2]$, so $u_{I_\varphi}(\varphi(x,y)) = \frac{1}{8n^2}(\langle t_1(x,y), t_2(x,y)\rangle + 4n^2)$. This time, let

$$\psi(x,y) = \frac{1}{2}\langle \frac{\lambda_1}{2n}x + \frac{\mu_1}{2n}y, \frac{\lambda_2}{2n} + \frac{\mu_2}{2n}y\rangle + \frac{1}{2}.$$

It is easily verified that this ψ is as desired.

For the induction step, suppose that $\varphi = u(\varphi_1, \ldots, \varphi_n)$, where

$$u : I_{\varphi_1} \times \cdots \times I_{\varphi_n} \to I_\varphi$$

is a surjective continuous function. By the induction hypothesis, there are $\mathcal{L}$-formulae $\psi_i(x)$ $(i = 1, \ldots, n)$ with each $I_{\psi_i} = [0,1]$ such that $H \models \sup_{\vec{x}} |u_{I_{\varphi_i}}(\varphi_i(\vec{x})) - \psi_i(\vec{x})| = 0$. Consider the $\mathcal{L}$-formula

$$\psi(x) = u_{I_\varphi}(u(u^{-1}_{I_{\varphi_1}}(\psi_1(\vec{x})), \ldots, u^{-1}_{I_{\varphi_n}}(\psi_n(\vec{x})))).$$

It is clear that $H \models \sup_x |u_\varphi(\varphi(\vec{x})) - \psi(\vec{x}))| = 0$. □

Corollary 4.6. *If $P : B_1(H)^n \to [0,1]$ is a uniformly continuous function, then P is an $\mathcal{L}$-definable predicate if and only if P is an $\mathcal{L}_{\mathrm{ms}}$-definable predicate*

Proof. This follows from the preceding corollary and the fact that the $\mathcal{L}_{ms}$-theory of H admits quantifier-elimination. □

Corollary 4.7. *Suppose that $f : H \to H$ is an $\mathcal{L}_{\mathrm{ms}}$-definable function such that $f(B_1(H)) \subseteq B_1(H)$. Then $f|B_1(H)$ is an $\mathcal{L}$-definable function.*

The definition of an $\mathcal{L}_{\mathrm{ms}}$-definable function is given in [10]. We should also remark that a similar discussion appears in [1].

Remark 4.8. It follows from the preceding corollary and Corollary 4.2 that for any $\mathcal{L}_{\mathrm{ms}}$-definable function $f : H \to H$, any $n \geq 1$, and any $\epsilon > 0$, there are scalars $\lambda_1, \ldots, \lambda_k$ and vectors $v_1, \ldots, v_k \in B_{m(n,f)}(H)$ such that, for all $x \in B_n(H)$, there is $i \in \{1, \ldots, k\}$ with $d(f(x), \lambda_i x + v_i) \leq \epsilon$. Using the main result of [10], we can give a different proof of this fact in the case that f is linear. Indeed, write $f = \lambda I + K$, where K is a compact operator. Let $\{v_1, \ldots, v_k\}$ be a finite ϵ-net for $K(B_n(H))$. Then for $a \in B_1(H)$, we have $d(K(a), v_i) \leq \epsilon$ for some $i \in \{1, \ldots, k\}$, whence $d(f(a), \lambda a + v_i) \leq \epsilon$. (Notice here that $\lambda_i = \lambda$ for all i.)

We now suppose that $\mathbb{K} = \mathbb{C}$ and set $\mathbb{S}^1 := \{\lambda \in \mathbb{C} \ : \ |\lambda| = 1\}$. We let $\mathcal{L}_U := \mathcal{L} \cup \{U, U^{-1}\}$, where U and U^{-1} are both unary function symbols. We let $T_U^{\forall}$ denote the $\mathcal{L}$-theory obtained from T by adding (universal) axioms saying that U is linear, preserves the inner product, and U and U^{-1} are inverses. (T_U axiomatizes the theory of an infinite-dimensional Hilbert space equipped with a unitary operator; one adds a symbol for U^{-1} so as to avoid the $\forall\exists$ axiom stating that U is onto.) We add to $T_U^{\forall}$ the following axioms:

$$\inf_x [|\langle x, x\rangle - 1| \dotplus d(Ux, \sigma x)|] = 0,$$

where σ ranges over a countable dense subset of $\mathbb{S}^1$. (These axioms assert that the spectrum of U is $\mathbb{S}^1$.) Then T_U is complete and admits quantifier elimination (see [3]); T_U is the theory of infinite-dimensional Hilbert spaces equipped with a generic automorphism. Since T_U is primitive, we can once again apply Proposition 3.4.

Lemma 4.9. *If $t(x)$ is an $\mathcal{L}_U(H)$-term, then there are $l, m \in \mathbb{Z}$, $l \leq m$, $\alpha_l, \ldots, \alpha_m \in \mathbb{D}$ and a vector $v \in B_1(H)$ such that, for all $a \in B_1(H)$, we have*

$$t(a) = v + \sum_{j=l}^{m} \alpha_j U^j(a).$$

Proof. This is proved by induction on the complexity of $t(x)$ exactly as in Lemma 4.1. □

Suppose that (H^*, U^*) is an elementary extension of (H, U).

Corollary 4.10. *Suppose that $f : H \to H$ is an $\mathcal{L}_U$-definable function and $\epsilon > 0$. Then there are $l, m \in \mathbb{Z}$, $l \leq m$, $\lambda_l^1, \ldots, \lambda_m^1, \ldots, \lambda_l^k, \ldots, \lambda_m^k \in \mathbb{D}$, and $v_1, \ldots, v_k \in B_1(H)$, such that, for all $a \in B_1(H^*)$, there is $i \in \{1, \ldots, k\}$ such that*

$$d(f(x), v_i + \sum_{j=l}^{m} \alpha_j^i U^j(x)) < \epsilon.$$

One can generalize this situation as follows: Let G be a countable (discrete) group and let $\mathcal{L}_G$ be the language for Hilbert spaces as above augmented by unary function symbols τ_g for $g \in G$. Let T_G be the universal $\mathcal{L}_G$-theory of a unitary representation of G on an infinite-dimensional Hilbert space. (As above, the axiom $\sup_x d((\tau_g(\tau_{g^{-1}}(x)), x) = 0$ allows us to assert that τ_g is onto without using a $\forall\exists$ axiom.) Let $\pi : G \to U(H)$ be a unitary representation of G on an (infinite-dimensional) Hilbert space H such that (H, π) is an existentially closed model of T_G (such an existentially closed model exists because T_G is an inductive theory). Let Σ be the set of existential consequences of (H, π). Then it is shown in [4] that $T_{GA} := T_G \cup \Sigma$ axiomatizes the class of existentially closed models of T_G, whence is the model companion of T_G. Moreover, since T_G has the amalgamation property (see [4]), it follows that T_{GA} admits quantifier elimination. As above, one can show that any $\mathcal{L}_G$ term $t(x)$ has the form $v + \sum_{i=1}^{n} \lambda_i g_i x$ for some $v \in B_1(H)$, some $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$, and some $g_1, \ldots, g_n \in G$. (Here we abuse notation and write gx instead of $\tau_g(x)$.) Consequently, we have:

Corollary 4.11. *Let (H, π) be any model of T_{GA} and let $f : H \to H$ be an $\mathcal{L}_G$-definable function. Then, for any $\epsilon > 0$, there are $v_1, \ldots, v_k \in B_1(H)$, scalars $\lambda_1^1, \ldots, \lambda_m^1, \ldots, \lambda_1^k, \ldots, \lambda_m^k \in \mathbb{D}$, and group elements $g_1, \ldots, g_k \in G$ such that, for all $a \in B_1(H^*)$, there is $i \in \{1, \ldots, k\}$ such that*

$$d(f(a), v_i + \sum_{j=1}^{m} \lambda_j^i g_j a) < \epsilon.$$

There is yet another expansion of Hilbert spaces that fits into this context. Let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$, where P is a new unary function symbol. We consider the theory T_P obtained from the theory of infinite-dimensional Hilbert spaces obtained by adding the following axioms (the latter two are axiom schemes, including one such axiom for every $n \geq 1$):

- P is linear;
- $\sup_x d(P^2(x), P(x)) = 0$;
- $\sup_{x,y} |\langle P(x), y\rangle - \langle x, P(y)\rangle| = 0$;
- $\inf_{v_1} \cdots \inf_{v_n} \max(\max_{i,j} |\langle v_i, v_j\rangle \dot{-} \delta_{ij}|, \max_i d(P(v_i), v_i))) = 0$;
- $\inf_{v_1} \cdots \inf_{v_n} \max(\max_{i,j} |\langle v_i, v_j\rangle \dot{-} \delta_{ij}|, \max_i d(P(v_i), 0))) = 0$.

The first three axioms say that P is a projection operator on H and the latter two axiom schemes say that $P(H)$ and $P(H)^{\perp}$ are infinite-dimensional. Then T_P is a complete theory with quantifier elimination ([5]); in fact, it is the theory of beautiful pairs of Hilbert spaces and its unique separable model is the Fraisse limit of the family of finite-dimensional Hilbert spaces equipped with projection operators.

Since T_P is a primitive theory with quantifier elimination, we may use Proposition 3.4. Let (H, P) be a model of T_P. Then in (H, P), all $\mathcal{L}$-terms $t(x)$ are easily seen to equivalent to terms be of the form $\alpha x + \beta P(x) + v$, where $\alpha, \beta \in \mathbb{D}$ and $v \in B_1(H)$. Thus:

Proposition 4.12. *Let $f : B_1(H) \to B_1(H)$ be an $\mathcal{L}_P$-definable function. Then for any $\epsilon > 0$, there are $v_1, \ldots, v_k \in B_1(H)$ and $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in \mathbb{D}$ such that, for all $a \in B_1(H)$, there is $i \in \{1, \ldots, k\}$ such that*

$$d(f(a), \alpha_i a + \beta_i P(a) + v_i) < \epsilon.$$

Consequently, for any elementary extension (H^*, P^*) of (H, P) and any $a \in B_1(H^*)$, there are $\alpha, \beta \in \mathbb{D}$ and $v \in B_1(H)$ such that $f(a) = \alpha a + \beta P^*(a) + v$.

Remark 4.13. The referee pointed out that the preceding examples are specific instances of a more general phenomenon: Given a fixed unital C^*-algebra A, one can consider the theory of presentations of A, that is, of Hilbert spaces H with a C^*-algebra morphism from A to $B(H)$, presented as Hilbert spaces with each $a \in A$ named as a unary function, and with universal axioms expressing that this is indeed a representation. The model completion of this theory can be obtained by adding some existential axioms essentially saying that the presentation is faithful.

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University of California, Los Angeles, Department of Mathematics, 520 Portola Plaza, Box 951555, Los Angeles, CA 90095-1555, USA

E-mail address: isaac@math.ucla.edu

URL: www.math.ucla.edu/~isaac