FURTHER THOUGHTS ON DEFINABILITY IN THE URYSOHN SPHERE

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Abstract. We discuss some basic geometry of sets definable in the Urysohn sphere using only finitely many parameters and briefly remark on the case of arbitrary definable sets. Then we discuss definable functions in the Urysohn sphere satisfying a special syntactic property.

1. introduction

Understanding the sets and functions definable in a given structure is a common goal in model theory. While there are adequate notions of definability for *metric structures* using a continuous version of first-order logic (see [1]), there has been very little study of what sets and functions are definable in concrete metric structures. The author has undertaken the task of trying to understand definable functions in various metric structures, namely the *Urysohn sphere* (the unique universal and ultrahomogeneous metric space of diameter at most 1) [3] and Hilbert spaces (and some of their generic expansions) [4], [5].

Describing the sets definable in a particular metric structure appears to be a much harder task. In this paper, we give an adequate description of the sets definable in the Urysohn sphere defined using only *finitely* many parameters (definable sets in continuous logic are generally allowed to use countably many parameters in their definition). We then proceed to explain some of the difficulties involved in describing arbitrary definable subsets of the Urysohn sphere.

In [3], a reasonable description of the functions definable in the Urysohn sphere was given, although a complete characterization is still lacking. In the final section of this paper, we show how to completely characterize the definable functions in the Urysohn sphere that satisfy a certain syntactic requirement.

We assume familiarity with continuous logic; the unacquainted reader can consult [1]. However, since this is an article about definable sets, we repeat the definition of a definable set in a metric structure.

Definition 1.1. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$.

(1) A continuous function $P : M^n \to [0,1]$ is an *A-definable predicate* if there are $\mathcal{L}(A)$-formulae $\varphi_n(x)$ such that the functions $\varphi_n^{\mathcal{M}}$:

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$M^n \to [0,1]$ converge uniformly to P. (Equivalently: there are $\mathcal{L}(A)$-formulae $\varphi_n(x)$ and a continuous function $u : [0,1]^{\mathbb{N}} \to [0,1]$ such that $P(x) = u((\varphi_n(x)))$ for all $x \in M^n$.)

(2) A closed set $X \subseteq M^n$ is *A-definable* if the function $d(x,X) : M^n \to [0,1]$ is A-definable.

We will use the following notation throughout this paper: $\mathfrak{U}$ denotes the Urysohn sphere, considered as a metric structure in the empty language consisting solely of the symbol d for the metric while $\mathbb{U}$ denotes an ω_1-saturated elementary extension of $\mathfrak{U}$. For $a \in \mathfrak{U}$ and $r \in [0,1]$, we set:

- $B(a;r) := \{x \in \mathfrak{U} \mid d(a,x) \leq r\}$, the *closed ball in* $\mathfrak{U}$ *centered at a with radius r*,
- $B^o(a;r) := \{x \in \mathfrak{U} \mid d(a,x) < r\}$, the *open ball in* $\mathfrak{U}$ *centered at a with radius r*, and
- $S(a;r) := \{x \in \mathfrak{U} \mid d(a,x) = r\}$, the *sphere in* $\mathfrak{U}$ *centered at a with radius r*.

At times, we will use the fact that, for $A \subseteq \mathfrak{U}$ (or $\mathbb{U}$), $\operatorname{dcl}(A) = \operatorname{acl}(A) = \overline{A}$, where $\overline{A}$ denotes the metric closure of A in $\mathfrak{U}$ (resp. $\mathbb{U}$); see [2, Fact 5.3] for a proof of this fact.

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2. Finitely Definable Sets

In this section, we prove some properties about A-definable subsets of $\mathfrak{U}$, where $A \subseteq \mathfrak{U}$ is finite. The key observation is the following (we thank Ward Henson for a useful discussion concerning the last implication).

Lemma 2.1. *Suppose that T is an ω-categorical (continuous) theory and that $\mathcal{M} \models T$. For a closed set $X \subseteq M^n$, the following are equivalent:*

(1) *X is type-definable over $\emptyset$;*
(2) *X is definable over $\emptyset$;*
(3) *X is a zeroset over $\emptyset$;*
(4) *X is fixed setwise by* $\operatorname{Aut}(\mathcal{M})$.

Proof. (1)$\Rightarrow$ (2) follows from ω-categoricity; see [1, Section 12]. (2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are always true. (4)$\Rightarrow$(1): Suppose that $c \in X$ and $\operatorname{tp}(c) = \operatorname{tp}(d)$. Since $\mathcal{M}$ is strongly ω-near-homogeneous ([1, Corollary 12.11]), there are, for $n \geq 1$, $\sigma_n \in \operatorname{Aut}(\mathcal{M})$ such that $\sigma_n(c) \to d$. Since each $\sigma_n(c) \in X$, we have $d \in \overline{X} = X$. It follows that $X = \bigcup_{p \in C} p(M^n)$ for some $C \subseteq S_n(T)$. (Here, $p(M^n)$ denotes the set of realizations of p in $\mathcal{M}$.) We claim that C is closed in the d-topology on $S_n(T)$. Indeed, suppose that p is in the d-closure of C and $a \models p$. Fix $\epsilon > 0$. Then there is $p' \in C$ such that $d(p,p') < \epsilon$. Thus, there are $a' \models p$ and $b \models p'$ such that $d(a',b) < \epsilon$. Since $b \in X$, we get that $d(a',X) \leq \epsilon$. However, $\operatorname{tp}(a) = \operatorname{tp}(a')$, so by invariance of X, we get that $d(a,X) \leq \epsilon$. Since X is closed, we get that $a \in X$, whence $p \in C$.

Since T is ω-categorical, C is closed in the *logic* topology on $S_n(T)$, whence there is a set $\Gamma(x_1, \ldots, x_n)$ of formulae such that $C = \{p \in S_n(T) \mid \Gamma \subseteq p\}$. It follows that X is type-defined by Γ. $\square$

Suppose that $A \subseteq \mathfrak{U}$ is relatively compact. Then, by compact homogeneity of $\mathfrak{U}$ (see [6, Section 4.5]), $\mathrm{Th}(\mathfrak{U}; (c_a)_{a \in A})$ is ω-categorical. Consequently, we have the following:

Corollary 2.2. *For relatively compact $A \subseteq \mathfrak{U}$ and closed $X \subseteq \mathfrak{U}^n$, the following are equivalent:*

(1) *X is type-definable over A;*
(2) *X is definable over A;*
(3) *X is a zeroset over A;*
(4) *X is fixed setwise by isometries of $\mathfrak{U}$ which fix A pointwise.*

In particular, the only $\emptyset$-definable sets in $\mathfrak{U}$ are $\emptyset$ and $\mathfrak{U}^n$.

The following lemma shows that certain topological and set-theoretic constructions preserve A-definability.

Lemma 2.3. *Suppose that $F, G \subseteq \mathfrak{U}$ are A-definable. Then:*

(1) *∂F is A-definable;*
(2) *$\overline{\mathrm{int}(F)}$ is A-definable;*
(3) *$\overline{\mathfrak{U} \setminus F}$ is A-definable.*
(4) *$F \cap G$ is A-definable.*
(5) *Write $F = P \sqcup C$, where P is the perfect kernel of F and C is the scattered part of F. Then P and $\overline{C}$ are A-definable.*

Proof. These are all immediate from A-invariance. $\square$

For $\vec{a} = (a_1, \ldots, a_n) \in \mathfrak{U}^n$ and $\vec{r} = (r_1, \ldots, r_n) \in [0, 1]^n$, set

$$S(\vec{a}; \vec{r}) := S(a_1; r_1) \cap \cdots \cap S(a_n; r_n).$$

Corollary 2.4. *Suppose that $F \subseteq \mathfrak{U}$ is closed. Set $A := \{a_1, \ldots, a_n\}$. Then F is A-definable if and only if there is a closed set $X \subseteq [0, 1]^n$ such that $F = \bigcup_{\vec{r} \in X} S(\vec{a}; \vec{r})$.*

Proof. The "if" direction follows from the characterization of definability in terms of invariance under isometries fixing A. For the "only if" direction, let $X = \{\vec{r} \in [0, 1]^n \mid \varphi(\vec{r}) = 0\}$, where $\varphi : [0, 1]^n \to [0, 1]$ is such that $d(x, F) = \varphi(d(x, a_1), \ldots, d(x, a_n))$. $\square$

Corollary 2.5. *$B(a; r)$ is A-definable if and only if $a \in A$.*

Proof. The "if" direction is clear. Now suppose that $B(a; r)$ is A-definable. Let ϕ be an isometry fixing A. Then ϕ is an isometry fixing $B(a; r)$, whence it fixes a. Thus $\{a\}$ is A-definable, implying that $a \in \mathrm{dcl}(A) = A$ since A is finite. $\square$

While the preceding corollary has a nice geometric proof, it is actually a special case of the following general result.

Lemma 2.6. *Suppose that F is A-definable. Then there is a finite $A_0 \subseteq A$ such that whenever F is B-definable, then $A_0 \subseteq \bar{B}$.*

Proof. This follows from the proof of the fact that $T_{\mathfrak{U}}$ has weak elimination of finitary imaginaries; see [2, Section 5]. $\square$

Define a *(closed) annulus* in $\mathfrak{U}$ to be a set of the form

$$\bar{A}(a; r_1, r_2) := \{x \in \mathfrak{U} \mid r_1 \leq d(x, a) \leq r_2\},$$

where $0 \leq r_1 \leq r_2 \leq 1$. (An open annulus $A(a; r_1, r_2)$ is defined similarly, strengthening the inequality signs.) We call a the *center* of the annulus. Note that a closed ball centered at a of radius r is an annulus centered at a (take $r_1 = 0$, $r_2 = r$) and a sphere centered at a of radius r is an annulus centered at a (take $r_1 = r_2 = r$). Also $\{a\}$ is an annulus centered at a (take $r_1 = r_2 = 0$). By Corollary 2.4, every annulus is definable over its center. Conversely, we have:.

Corollary 2.7. *If F is a nonempty, connected $\{a\}$-definable subset of $\mathfrak{U}$, then F is a closed annulus centered at a.*

Proof. By Corollary 2.4, there is a nonempty closed $X \subseteq [0, 1]$ such that

$$F = \bigcup_{r \in X} S(a; r).$$

We must show that X is a closed subinterval of $[0, 1]$, that is we must show that X is convex. Suppose that $0 \leq r_1 < s < r_2 \leq 1$, where $r_1, r_2 \in X$. Suppose that $s \notin X$. Then $B^o(a; s) \cap F$ and $B(a; s)^c \cap F$ yield a disconnection of F, a contradiction. $\square$

Fix an annulus $A := A(a; r, R)$. For $x \in A$, define the *local diameter of A at x* to be $\operatorname{diam}(x) := \sup\{d(x, y) \mid y \in A\}$. Define the *radius* of A to be the quantity $\inf\{\operatorname{diam}(x) \mid x \in A\}$.

Proposition 2.8. *For an annulus $A = A(a; r, R)$, the diamter of A is $2R$ and the radius of A is $r + R$.*

Proof. Since $d(a, x) = R$, $d(a, y) = R$, and $d(x, y) = 2R$ defines a metric space, it can be realized in $\mathfrak{U}$, whence $\operatorname{diam}(A) \geq 2R$. However, the triangle inequality yields $\operatorname{diam}(A) \leq 2R$, whence $\operatorname{diam}(A) = 2R$.

Next, suppose that $d(x, a) = r$. Then since we can realize $d(x, a) = r$, $d(y, a) = R$, $d(x, y) = r + R$ inside of $\mathfrak{U}$, we see that $\operatorname{diam}(x) \geq r + R$. However, for all $z \in \mathfrak{U}$, $d(x, z) \leq d(x, a) + d(a, z) \leq r + R$. Thus, $\operatorname{diam}(x) = r + R$, whence the radius of A is bounded above by $r + R$. By embedding an annulus in the euclidean plane of inner and outer radii r and R respectively in $\mathfrak{U}$, we see that $\operatorname{diam}(x) \geq r + R$ for each $x \in A$, whence the radius of A is at least $r + R$. $\square$

Corollary 2.9. *Two annuli $A(a; r, R)$ and $A(a'; r', R')$ are isometric if and only if $r = r'$ and $R = R'$.*

Consequently, we see that the class of definable sets (even over one element) is quite exotic in the sense that there are continuum many non-isometric definable sets.

Define a *generalized annulus centered around* $(a_1, \ldots, a_n)$ to be a set of the form $\bigcup_{\vec{r} \in X} S(\vec{a}; \vec{r})$ where X is a nonempty sub*continuum* of $[0,1]^n$. The following is a generalization of Corollary 2.7.

Corollary 2.10. *If F is a nonempty connected A-definable subset of $\mathfrak{U}$, where $A = \{a_1, \ldots, a_n\}$, then F is a generalized annulus centered around $(a_1, \ldots, a_n)$.*

Proof. Write $F = \bigcup_{\vec{r} \in X} S(\vec{a}; \vec{r})$. Without loss of generality, we may suppose that $S(\vec{a}; \vec{r}) \neq \emptyset$ for each $\vec{r} \in X$. We will show that X is connected. Suppose, towards a contradiction, that there are disjoint open $\mathrm{O}_1, \mathrm{O}_2 \subseteq \mathbb{R}^n$ such that $X = (X \cap \mathrm{O}_1) \cup (X \cap \mathrm{O}_2)$. For $i = 1, 2$, set $F_i := \bigcup_{\vec{r} \in X \cap \mathrm{O}_i} S(\vec{a}; \vec{r})$. Clearly each $F_i \neq \emptyset$ and $F = F_1 \sqcup F_2$. We must show that each F_i is open in F. Fix $i \in \{1, 2\}$ and write $\mathrm{O}_i := \bigcup_\alpha \prod_{j=1}^n (b_j^\alpha, c_j^\alpha)$. Fix $y \in F_i$ and take α such that $(d(y, a_1), \ldots, d(y, a_n)) \in X \cap \prod_{j=1}^n (b_j^\alpha, c_j^\alpha)$. Fix $\epsilon > 0$ small enough such that $(d(y, a_j) - \epsilon, d(y, a_j) + \epsilon) \subseteq (b_j^\alpha, c_j^\alpha)$ for each $j = 1, \ldots, n$. It follows that if $z \in F$ is such that $d(y, z) < \epsilon$, then $z \in F_i$. $\square$

We now consider the decomposition of definable sets into their connected components. For the rest of this section, we assume that $A \subseteq \mathfrak{U}$ is finite.

Lemma 2.11. *Suppose that F is an A-definable set and that C is a connected component of F. Then C is A-definable.*

Proof. Fix $c \in C$. Then since every isometry of $\mathfrak{U}$ fixing Ac pointwise fixes C setwise, we have that C is Ac-definable. Suppose that there is $d \in C \setminus \{c\}$. Then C is Ad-definable. It follows that $d(x, C)$ is both Ac-definable and Ad-definable. Since $T_{\mathfrak{U}}$ admits weak elimination of finitary imaginaries, we have that $d(x, C)$ is A-definable.

It remains to show that every connected component of cardinality 1 is A-definable. Set

$$F' := \{c \in F \mid \{c\} \text{ is a connected component of } F\}.$$

Then F' is A-invariant, whence $\overline{F'}$ is A-definable. Thus, there is $X \subseteq [0,1]^n$ such that

$$\overline{F'} = \bigcup_{\vec{r} \in X} (S(a_1; r_1) \cap \cdots \cap S(a_n; r_n)).$$

Fix $c \in F'$. Choose $\vec{r} \in X$ such that $c \in S := S(a_1; r_1) \cap \cdots \cap S(a_n; r_n)$. Since spheres in the Urysohn space are connected (see [6, Section 4.3]), we have $\{c\} = S$, whence $\{c\}$ is A-definable. $\square$

Corollary 2.12. *A Cantor set in $\mathfrak{U}$ cannot be A-definable.*

More generally:

Corollary 2.13. *If X is an A-definable compact set, then X is a finite subset of A.*

Proof. The connected components are A-definable, so generalized annuli. However, they are also compact and the only compact generalized annuli are one-element subsets of A (as compact subsets of $\mathfrak{U}$ have no interior). $\square$

The next result says that, for sets defined over finitely many parameters, if there are infinitely many connected components, then the connected components cannot be a uniform distance from one another. For a subset C of $\mathfrak{U}$ and $\epsilon > 0$, we let $N(C, \epsilon)$ denote the open ϵ-neighborhood around C.

Corollary 2.14. *Suppose that F is A-definable. Then, given any $\epsilon > 0$, there are finitely many connected components $C_1, \ldots, C_n$ of F such that $F \subseteq \bigcup_{i=1}^n N(C_i; \epsilon)$.*

Proof. For $n \geq 1$, let $\varphi_n(x)$ be a formula with parameters from A such that $|d(x, F) - \varphi_n(x)| < \frac{1}{n}$ for all $x \in \mathfrak{U}$.

Fix $\epsilon > 0$ and let $(C_i \mid i < \alpha)$ enumerate the connected components of F. Since all of the predicates $d(x, C_i)$ are A-definable, we may find formulae $\psi_n^i(x)$ with parameters from A such that $|d(x, C_i) - \psi_n^i(x)| < \frac{1}{n}$. Fix $m \in \mathbb{N}$ such that $\frac{2}{m} < \epsilon$. Since the set of conditions

$$\{\varphi_n(x) \leq \frac{1}{n} \mid n \geq 1\} \cup \{\psi_n^i(x) \geq \frac{1}{n} \mid i < \alpha, n \geq m\}$$

is unsatisfiable, by ω-saturation, there are $i_1, \ldots, i_k < \alpha$ and $n_1, \ldots, n_k \geq m$ such that

$$\{\varphi_n(x) \leq \frac{1}{n} \mid n \geq 1\} \cup \{\psi_{n_j}^{i_j}(x) \geq \frac{1}{n_j} \mid j = 1, \ldots, k\}$$

is unsatisfiable, yielding the desired result. $\square$

3. Arbitrary Definable Sets

Of course, being definable over a finite set of parameters is a very special thing in continuous logic. We would thus like to have some results concerning subsets of $\mathfrak{U}$ (or $\mathbb{U}$) defined over countably many parameters. The following example shows that our characterization of definability over finite parametersets fails for sets defined over a countably infinite set of parameters.

Example 3.1. Suppose that $A = \{a_n \mid n \in \mathbb{N}\}$ is a countable subset of $\mathbb{U}$ such that $d(a_i, a_j) = 1$ for all distinct $i, j \in \mathbb{N}$. Set

$$F := \{x \in \mathbb{U} \mid d(x, A) \geq \frac{1}{2}\},$$

a closed, A-invariant set. We claim that F is *not* A-definable. Indeed, if F were A-definable, then there would be a continuous function $\varphi : [0, 1]^{\mathbb{N}} \to [0, 1]$ such that $d(x, F) = \varphi((d(x, a_i)))$ for all $x \in \mathbb{U}$. Fix $\epsilon \in (0, \frac{1}{3})$. Choose $\delta > 0$ and $n \in \mathbb{N}^{>0}$ such that, for all $\vec{w}, \vec{z} \in [0, 1]^{\mathbb{N}}$, if $|w_i - z_i| < \delta$ for all $i < n$, then $|\varphi(\vec{w}) - \varphi(\vec{z})| < \epsilon$. Now take $x, y \in \mathbb{U}$ such that $d(x, a_i) = d(y, a_i) = 1$

for all $i \in \mathbb{N} \setminus \{n\}$ while $d(x, a_n) = \frac{1}{3}$ and $d(y, a_n) = \frac{1}{3} - \epsilon$. By the choice of n, we see that $|d(x, F) - d(y, F)| < \epsilon$. However, $d(x, F) = \frac{1}{2} - \frac{1}{3}$ and $d(y, F) = \frac{1}{2} - \frac{1}{3} + \epsilon$, a contradiction.

In all actuality, there is probably very little hope of classifying all definable subsets of $\mathfrak{U}$. Indeed, in any metric structure $\mathcal{M}$, any compact subset of M is definable (see [1, Proposition 9.19]). Since every compact metric space of diameter ≤ 1 embeds in $\mathfrak{U}$, it follows that *every compact metric space of diameter* ≤ 1 *is isometric to a definable subset of* $\mathfrak{U}$.

That being said, we would like to point out that understanding definable subsets of $\mathbb{U}$ will sometimes allow us to prove facts about definable subsets of $\mathfrak{U}$. The idea is to use the "canonical extension" of a definable subset of $\mathfrak{U}$ to a definable subset of $\mathbb{U}$. Since this canonical extension notion does not really appear in the literature, we now discuss it in more detail.

Suppose that $F \subseteq \mathfrak{U}$ is A-definable, where $A \subseteq \mathfrak{U}$ is countable. Then $P(x) := d(x, F)$ is an A-definable predicate in $\mathfrak{U}$. We know that there is a unique A-definable predicate Q in $\mathbb{U}$ extending P. Moreover, we have $(\mathfrak{U}, P) \preceq (\mathbb{U}, Q)$. (See [1, Theorem 9.8].) Now, since P is a distance predicate, it satisifes axioms (E1) and (E2) of [1, Section 9], whence by elementarity, Q satisfies (E1) and (E2). Let $\tilde{F} := \{x \in \mathbb{U} \mid Q(x) = 0\}$. By [1, Theorem 9.12], $Q(x) = d(x, \tilde{F})$. It follows that $F \subseteq \tilde{F}$, $\tilde{F}$ is A-definable and $\tilde{F} \cap \mathfrak{U} = A$.

Conversely, suppose that $E \subseteq \mathbb{U}$ is A-definable, where $A \subseteq \mathfrak{U}$ is countable. We claim that $E \cap \mathfrak{U}$ is an A-definable subset of $\mathfrak{U}$. Let $Q : \mathbb{U} \to [0, 1]$ be the A-definable predicate given by $Q(x) = d(x, E)$. Let $Q = P \restriction \mathfrak{U}$, so $(\mathfrak{U}, P) \preceq (\mathbb{U}, Q)$. Then $Z(P)$ is an A-definable subset of $\mathfrak{U}$; but $Z(P) = Z(Q) \cap \mathfrak{U} = E \cap \mathfrak{U}$.

Corollary 3.2. *Suppose that $F, G \subseteq \mathfrak{U}$ are A-definable, where A is a countable subset of $\mathfrak{U}$. Suppose that $\tilde{F} \cap \tilde{G}$ is A-definable. Then $F \cap G$ is A-definable.*

Proof. $F \cap G = (\tilde{F} \cap \tilde{G}) \cap \mathfrak{U}$ is A-definable. $\square$

Along these same lines:

Corollary 3.3. *If $F \subseteq \mathfrak{U}$ is A-definable, where $A \subseteq \mathfrak{U}$ is countable, and the perfect kernel of $\tilde{F}$ is A-definable, then the perfect kernel of F is A-definable.*

Proof. The perfect kernel of $\tilde{X}$ is A-definable and the intersection of the perfect kernel of $\tilde{X}$ with $\mathfrak{U}$ is the perfect kernel of X. $\square$

Corollary 3.4. *Suppose $F \subseteq \mathfrak{U}$ is A-definable, where $A \subseteq \mathfrak{U}$ is countable. Then $\overline{\mathbb{U} \setminus \tilde{F}} \cap \mathfrak{U} = \overline{\mathfrak{U} \setminus F}$. Consequently, $\overline{\mathfrak{U} \setminus F}$ and ∂F are A-definable if $\overline{\mathbb{U} \setminus \tilde{F}}$ is A-definable.*

Proof. It is clear that $\overline{\mathfrak{U} \setminus F} \subseteq \overline{\mathbb{U} \setminus \tilde{F}} \cap \mathfrak{U}$. We now prove the other direction. Suppose that $x \in \overline{\mathbb{U} \setminus \tilde{F}} \cap \mathfrak{U}$. Let $x_n \in \mathbb{U} \setminus \tilde{F}$ be such that $d(x_n, x) \leq \frac{1}{n}$. Set

$\epsilon_n := d(x_n, \tilde{F}) > 0$. Then, for every n, we have

$$\mathbb{U} \models \inf_z \max(\epsilon_n \dot{-} Q(z), d(z,x) \dot{-} \frac{1}{n}) = 0.$$

By elementarity, the above condition is true in $\mathfrak{U}$, with $P(z)$ replacing $Q(z)$. Take $0 < \delta_n < \min(\epsilon_n, \frac{1}{n})$. Then there is $z_n \in \mathfrak{U}$ such that

$$\max(\epsilon_n \dot{-} P(z_n), d(z_n, x) \dot{-} \frac{1}{n}) \leq \delta_n.$$

Note then that $P(z_n) \geq \epsilon_n - \delta_n > 0$, whence $z_n \in \mathfrak{U} \setminus F$, and $d(z_n, x) \leq \frac{2}{n}$. It follows that $x \in \overline{\mathfrak{U} \setminus F}$. $\square$

4. Special Definable Functions

In [3], the following theorem on definable functions in $\mathfrak{U}$ was proven:

Theorem 4.1. *If $f : \mathfrak{U}^n \to \mathfrak{U}$ is an A-definable function, where $A \subseteq \mathfrak{U}$ is countable, then either f is a projection function (namely, there is $i \in \{1, \ldots, n\}$ such that, for all $x = (x_1, \ldots, x_n) \in \mathfrak{U}^n$, $f(x) = x_i$) or else $f(\mathfrak{U}^n)$ is a relatively compact subset of $\overline{A}$.*

While this theorem can be used to draw many interesting corollaries, it still doesn't provide an exact characterization of the definable functions in $\mathfrak{U}$. In [3], the following conjecture appeared.

Conjecture 4.2. *If $f : \mathfrak{U}^n \to \mathfrak{U}$ is an A-definable function, where $A \subseteq \mathfrak{U}$ is countable, then either f is a projection function or else f is constantly equal to an element of $\bar{A}$.*

In [3], it is shown that if the conjecture is true for one-variable definable functions, then it is true for all definable functions.

While this conjecture remains open, it is the goal of this section to prove that the conjecture holds under a (strong) syntactic constraint on the definable functions. Indeed, the author's initial approach to studying definable functions in $\mathfrak{U}$ (which ultimately did not work) was to use the fact that the predicate $d(f(x), y)$ was approximable by formulae. More precisely, by quantifier elimination for $\mathrm{Th}(\mathfrak{U})$, for every $n \geq 2$, there is a quantifier-free *restricted* formula $\varphi_n(x, y)$ with parameters from A such that

$$|d(f(x), y) - \varphi_n(x, y)| \leq 2^{-n}$$

for all $x, y \in \mathfrak{U}$. (See [1, Section 6] for the definition of a restricted formula.) Although in general we could not make this approach work, we can make it work for special kinds of φ_n:

Definition 4.3. We define what it means for a formula $\varphi(\vec{x})$ to be a **generalized atomic formula** by induction:

- If φ is atomic, then it is generalized atomic.
- If φ is generalized atomic, then so is $\frac{1}{2}\varphi$.
- Nothing else is a generalized atomic formula.

Definition 4.4. We define what it means for a formula $\varphi(\vec{x})$ to be a **special restricted formula** by induction:

- If φ is a generalized atomic formula, then it is a special restricted formula.
- If φ is a special restricted formula and ψ is a generalized atomic formula, then $\varphi \dot{-} \psi$ is a special restricted formula.
- Nothing else is a special restricted formula.

Remark 4.5. Note that if φ is a special restricted formula, then so is $\frac{1}{2}\varphi$ (not literally, but rather up to logical equivalence). This follow by induction and the fact that $\frac{1}{2}(a \dot{-} b) = \frac{1}{2}a \dot{-} \frac{1}{2}b$. Thus the only difference between a special restricted formula and a restricted formula is the way one is allowed to use the $\dot{-}$ connective.

Call a definable predicate P **A-special** if it can be approximated by special restricted formula with parameters from A plugged in. Likewise, call a definable function $f : \mathfrak{U}^n \to \mathfrak{U}$ **A-special** if the definable predicate $d(f(x), y)$ is A-special.

Proposition 4.6. *If $f : \mathfrak{U} \to \mathfrak{U}$ is an A-special definable function, then f is either the identity function or constantly equal to an element of $\bar{A}$.*

Proof. For $n \geq 2$, we let $\varphi_n(x, y)$ be a special restricted formula with parameters from A satisfying

$$|d(f(x), y) - \varphi_n(x, y)| \leq 2^{-n}$$

for all $x, y \in \mathfrak{U}$. Fix $n \geq 2$. Since φ_n is special, there are generalized atomic formulae $\psi_1(x, y), \ldots, \psi_m(x, y)$ with parameters from A such that

$$\varphi_n(x, y) = (\cdots((\psi_1(x, y) \dot{-} \psi_2(x, y)) \dot{-} \psi_3(x, y)) \cdots) \dot{-} \psi_m(x, y).$$

From the identity $(a \dot{-} b) \dot{-} c = a \dot{-} (b + c)$, we see that we we have

$$\varphi_n(x, y) = \psi_1(x, y) \dot{-} (\psi_2(x, y) + \cdots + \psi_n(x, y))$$

for all $x, y \in \mathfrak{U}$.

First suppose that $\psi_1(x, y) = 2^{-k} d(x, a)$ for some $k \geq 0$ and some $a \in A$. Then $\varphi_n(a, y) = 0$ for all $y \in \mathfrak{U}$, implying that $d(f(a), y) \leq 2^{-n}$ for all $y \in \mathfrak{U}$, which is a contradiction. Thus this case is impossible.

Next suppose that $\psi_1(x, y) = 2^{-k} d(y, a)$ for some $k \geq 0$ and some $a \in A$. Then $\varphi_n(x, a) = 0$ for all $x \in \mathfrak{U}$, implying that $d(f(x), a) \leq 2^{-n}$ for all $x \in \mathfrak{U}$, i.e. that $\text{image}(f) \subseteq B(a; 2^{-n})$.

Next suppose that $\psi_1(x, y) = 2^{-k} d(x, y)$ for some $k \geq 0$. Then $\varphi_n(x, x) = 0$ for all $x \in \mathfrak{U}$, implying that $d(f(x), x) \leq 2^{-n}$ for all $x \in \mathfrak{U}$.

Finally suppose that $\psi_1(x, y)$ has no occurrences of x or y. Then ψ_1 is either the constant 0 or the constant 2^{-k} for some $k \geq 1$ or the constant $2^{-k} d(a, a')$ for some $a, a' \in A$ and some $k \geq 0$. This case requires some work.

Let us denote $\psi_1(x,y)$ by the constant c. We may rewrite $\psi_2+\cdots+\psi_m$ as

$$\sum_{i=1}^{p} 2^{-l_i} d(x,y) + \sum_{j=1}^{q} 2^{-m_j} d(x,a_j) + \sum_{k=1}^{r} 2^{-n_k} d(y,b_k) + s,$$

where $l_i, m_j, n_k \geq 0$ and s is a constant which appears by summing together those ψ_i's which have no x or y in them. Choose $x_0 \in \mathfrak{U}$ such that $d(x_0,a_j)=1$ for all $j=1,\ldots,q$. Choose $y_0 \in \mathfrak{U}$ such that $d(x_0,y_0)=1$, $d(f(x_0),y_0)=1$, and $d(y_0,b_k)=1$ for all $k=1,\ldots,r$. Then

$$\varphi_n(x_0,y_0) = c \dot{-} (\sum_{i=1}^{p} 2^{-l_i} + \sum_{j=1}^{q} 2^{-m_j} + \sum_{k=1}^{r} 2^{-n_k} + s) =: c',$$

implying that $c' \geq 1-2^{-n}$. However, $\varphi_n(x,y) \geq c'$ for all $x,y \in \mathfrak{U}$, implying that $d(f(x),y) \geq 1-2^{-n+1}$ for all $x,y \in \mathfrak{U}$, which is a contradiction. Thus this case is impossible.

To summarize, by knowing that φ_n approximates $d(f(x),y)$ up to an error of 2^{-n}, we learn that either image$(f) \subseteq B(a;2^{-n})$ for some $a \in A$ or that $\|f - \mathrm{id}_{\mathfrak{U}}\|_\infty \leq 2^{-n}$. Let us call these options $(\mathrm{I})_n$ and $(\mathrm{II})_n$. If option $(\mathrm{I})_n$ happens for infinitely many n, then we see that f must be constantly equal to an element of $\bar{A}$. If option $(\mathrm{II})_n$ happens for infinitely many n, then $f=\mathrm{id}_{\mathfrak{U}}$. □

Corollary 4.7. *If $f:\mathfrak{U}^n \to \mathfrak{U}$ is an A-special definable function, then either f is a projection function or constantly equal to an element of $\bar{A}$.*

Proof. One proves this by induction on n, mimicking the proof of [3, Proposition 4.8], using Proposition 4.6 above to cover the base case. One will need to use the observation that if $f:\mathfrak{U}^n \to \mathfrak{U}$ is A-special, then the functions f_b and f^c are Ab-special and Ac-special respectively. □

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