CONNECTING H5 WITH IND-DEFINABLE GROUPS

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Let G be a locally compact, second countable group and let G^* be its nonstandard extension. We let μ and $\mu(0)$ denote the monad of the identity in G^* and the monad of 0 in $\mathbb{R}^*$ respectively. Fix $\sigma \in \mathbb{N}^* \setminus \mathbb{N}$ and let $\epsilon := \frac{1}{\sigma}$. Fix a decreasing sequence $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$ of open neighborhoods of the identity shrinking to the identity. Further suppose that for every $k, m, n \in \mathbb{N}$ and every $a \in G$, if m is even, $a^m \in U_k$ and $n \leq m$, then $a^n \in U_k$. (This happens, for example, when $\exp : V \to U_1$ is a homeomorphism, V is a balanced neighborhood of 0, and we set $U_k := \exp(\frac{1}{k}V)$.)

Lemma 0.1. *$G(\sigma)$ is a $\bigvee$-definable set.*

Proof. Recall that $G(\sigma) := \{a \in \mu \ : \ a^i \in \mu \text{ whenever } \epsilon \cdot i \in \mu(0)\}$. For each $n \in \mathbb{N}^{>0}$, let $i_n \in \mathbb{N}^*$ be divisible by all standard powers of 2 and satisfy $\epsilon \cdot i_n \in (\frac{1}{n+1}, \frac{1}{n})^*$. Observe now that, for each $n \in \mathbb{N}^{>0}$, we have

$$\{i \in \mathbb{N}^* \ : \ \epsilon \cdot i \in \mu(0)\} = \{i \in \mathbb{N}^* \ : \ i < \frac{i_n}{2^k} \text{ for all } k \in \mathbb{N}\}.$$

Now suppose that $a^{i_n} \in U_1^*$ for some $n \in \mathbb{N}^{>0}$. Then $a^{\frac{i_n}{2^k}} \in U_k^*$ for each $k \in \mathbb{N}$. By the transfer of our assumption, $a^i \in U_k^*$ for each k. Consequently, $a^i \in \mu$ for each $i \in \mathbb{N}^*$ satisfying $\epsilon \cdot i \in \mu(0)$, whence $a \in G(\sigma)$.

Conversely, suppose that $a \in G(\sigma)$. Then the partial type

$$\{\epsilon \cdot i < \frac{1}{n} \ : \ n \in \mathbb{N}\} \cup \{a^i \notin U_1^*\}$$

is inconsistent. By compactness, there is $n \in \mathbb{N}$ such that $\epsilon \cdot i < \frac{1}{n}$ implies that $a^i \in U_1^*$. It follows that $a^{i_n} \in U_1^*$. Thus, if we set

$$G_n := \{a \in G^* \ : \ a^{i_n} \in U_1^*\},$$

then it follows that $G(\sigma) = \bigcup_n G_n$. □

Remark 0.2. Part of the nonstandard proof of the H5 shows that $G(\sigma)$ is a group when G has NSS. Do we have $G_n^2 \subseteq G_{n+1}$? If so, then $G(\sigma)$ would be a $\bigvee$-definable group.

Let $\mathbb{R}_f$ denote the finite elements of $\mathbb{R}^*$.

Lemma 0.3. *$G^0(\sigma)$ is a $\bigwedge$-definable subset of $G(\sigma)$.*

Proof. Recall that $G^0(\sigma) = \{a \in \mu \ : \ a^i \in \mu \text{ whenever} \epsilon \cdot i \in \mathbb{R}_f\}$. Then set

$$H_n := \{a \in G^* \ : \ (\forall i \in \mathbb{N}^*)(i\epsilon < n \to a^i \in U_n^*)\}.$$

It is easy to see that $G^0(\sigma) = \bigcap_n H_n$.

$\square$