Nonstandard hulls of locally uniform groups

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Nonstandard hulls of uniform spaces á la Luxembourg

- Suppose that $(X, \mathcal{U})$ is a uniform space and $e \in X$.
- Let P be a family of pseudometrics generating the uniformity.
- Set $X_{\text{fin},P} := \{x \in X^* \mid p(x,e) \in \mathbb{R}_{\text{fin}} \text{ for all } p \in P\}$.
- Set $\mu(\mathcal{U}) := \bigcap_{U \in \mathcal{U}} U^*$.
- Set $\hat{X}_P := X_{\text{fin},P}/\mu(\mathcal{U})$.
- Then $\hat{X}_P$ is a uniform space with uniformity generated by the set of pseudometrics $\{\hat{p} \mid p \in P\}$, where $\hat{p}([x],[y]) := \text{st}(p(x,y))$. We refer to $\hat{X}_P$ as a *nonstandard hull* of X.

Remark

This procedure depends on the choice of P. For example, given P, define $P' := \{\min(p,1) \mid p \in P\}$. Then $X_{\text{fin},P'} = X^*$.

The case of topological groups

Let (G, τ) be a topological group. Then $(G, \mathcal{U})$ is a uniform space, where $\mathcal{U}$ is either the *left uniformity* $\mathcal{U}_l$ or the *right uniformity* $\mathcal{U}_r$:

1. $\mathcal{U}_l$ has base $\{(x, y) \mid x^{-1}y \in U\}$, with $U \in \tau$.
2. $\mathcal{U}_r$ has base $\{(x, y) \mid xy^{-1} \in U\}$, with $U \in \tau$.

Question

Is the nonstandard hull of $(G, \mathcal{U})$, in a natural way, a topological group again?

Answer

For certain groups, the answer is almost yes.

Locally uniform groups

$U^n := \{x_1 \cdots x_n \mid \text{ each } x_i \in U\}$.

Definition (Enflo)

G is *locally uniform* if there is a symmetric open neighborhood U of the identity in G and a uniformity $\mathcal{U}$ compatible with the topology such that $(x, y) \mapsto xy : U^2 \times U^2 \to U^4$ is $\mathcal{U}$-uniformly continuous.

Lemma (Enflo)

Suppose that G is locally uniform as witnessed by U and $\mathcal{U}$. Then $\mathcal{U}|U = \mathcal{U}_l|U = \mathcal{U}_r|U$ and $x \mapsto x^{-1} : U \to U$ is $\mathcal{U}$-uniformly continuous.

We will refer to G as being U-locally uniform. If $U = G$, then we call G a *uniform group*.

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Examples of locally uniform groups

Examples

1. Locally compact groups
2. Locally abelian groups
3. Groups that admit locally two-sided invariant metrics
4. Certain infinite-dimensional Lie groups

Equivalent formulations

Let P (resp. Q) be the set of left-invariant (resp. right-invariant) continuous pseudometrics on G.

Lemma

The following are equivalent:

1. *G is U-locally uniform;*
2. $\mathcal{U}_l|U = \mathcal{U}_r|U$;
3. $\mu(\mathcal{U}_l) \cap (U^* \times U^*) = \mu(\mathcal{U}_r) \cap (U^* \times U^*)$;
4. *for all $x, y \in U^*$: $x^{-1}y \in \mu \Leftrightarrow xy^{-1} \in \mu$;*
5. *μ is "normal" in U^*: for all $x \in U^*$ and $y \in \mu$, we have $xyx^{-1} \in \mu$;*
6. *for all $x, y \in U^*$:*

$$p(x, y) \approx 0 \text{ for all } p \in P \Leftrightarrow q(x, y) \approx 0 \text{ for all } q \in Q.$$

Nonstandard hulls

- Suppose that G is U-locally uniform, where U is symmetric, and let P be a family of left-invariant pseudometrics on G generating $\mathcal{U}_l$.
- Set $U_{\text{fin},P} := \{x \in U^* \mid \mu(x) \subseteq U^* \text{ and } p(x) \in \mathbb{R}_{\text{fin}} \text{ for all } p \in P\}$. If $U = G$, then this is Luxembourg's notion of finite.
- Set $\hat{U}_P := U_{\text{fin},P}/\mu$, naturally a uniform space just as before.
- Set $\Omega := \{([x],[y]) \in \hat{U}_P \times \hat{U}_P \mid xy \in U_{\text{fin},P}\}$.
- Then $(\hat{U}_P, \Omega)$ is a *local group* with respect to the topology inherited from the uniform structure and the map $x \mapsto [x] : G|U \to \hat{U}_P$ is an injective morphism of local groups.

Local Groups

Definition

A **local group** is a tuple $(H, 1, \iota, p)$ where:

- H is a hausdorff topological space with distinguished element $1 \in H$;
- $\iota : \Lambda \to H$ is continuous, where $\Lambda \subseteq H$ is open;
- $p : \Omega \to H$ is continuous, where $\Omega \subseteq H \times H$ is open;
- $1 \in \Lambda$, $\{1\} \times H \subseteq \Omega$, $H \times \{1\} \subseteq \Omega$;
- $p(1, x) = p(x, 1) = x$;
- if $x \in \Lambda$, then $(x, \iota(x)) \in \Omega$, $(\iota(x), x) \in \Omega$, and
$$p(x, \iota(x)) = p(\iota(x), x) = 1;$$
- if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then
$$p(p(x, y), z) = p(x, p(y, z)).$$

The extremes

- If P is the set of all left-invariant continuous pseudometrics on G, then $\hat{U}_P$ is the "smallest" nonstandard hull: for any P', the map $[x]_P \to [x]_{P'} : \hat{U}_P \to \hat{U}_{P'}$ is an injective morphism of local groups.
- If $P_1 := \{\min(p, 1) \mid p \in P\}$, then $\hat{U}_{P_1}$ is the "biggest" nonstandard hull: for any P', the map $[x]_{P'} \to [x]_{P_1} : \hat{U}_{P'} \to \hat{U}_{P_1}$ is an injective morphism of local groups.

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Metrizable groups

- Suppose that G is a metrizable group and you consider the set $\{d\}$, where d is a left-invariant metric. Suppose that G is U-locally uniform. Then $\hat{U}_d$ is metrizable.
- If d' is another left-invariant metric on G, then one can show that $\hat{U}_d$ and $\hat{U}_{d'}$ are locally isomorphic. Thus, there is a unique *metric nonstandard hull group germ*.

Theorem (G.)

Let MGrp *denote the category of metrizable groups and let* LocGrp *denote the category of local group germs. Then the metric nonstandard hull construction is a functor* MGrp $\rightarrow$ LocGrp.

Mal'cev hulls

- Suppose that H is a local group. Let H^M be the quotient of the set of words on H modulo the transitive closure of the "natural" contraction and expansion operations on words.
- Then H^M is naturally a group, called the *Mal'cev hull* of H.
- H^M satisfies the obvious universal mapping property for local group morphisms from H into topological groups.

Theorem (Mal'cev; van den Dries-G.)

If H is globally associative, then the canonical map $H \to H^M$ is injective.

Global nonstandard hulls

- It is easy to see that the nonstandard hull $\hat{U}$ is globally associative.
- Thus, we have the Mal'cev hull of $\hat{U}$, which we will call $\hat{G}$, the *global nonstandard hull of G*.
- What is the relationship between G and $\hat{G}$?
- Let U^M be the Mal'cev hull of U (as a local group). Then we have a unique injective group morphism $U^M \to \hat{G}$.
- Thus, if the natural map $U^M \to G$ is an isomorphism, we then have that G embeds as a subgroup of $\hat{G}$.
- **Fact (van den Dries):** If G is locally path connected and simply connected and U is connected, then $U^M \to G$ is an isomorphism.
- In general, the Mal'cev hulls of $\hat{U}$ and $\hat{V}$ will be non-isomorphic, so the global nonstandard hull of G is *non-canonical*.

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UNSS groups

Recall that G is *NSS* if there is a neighborhood U of 1 in G such that the only subgroup of G contained in U is $\{1\}$; equivalently, the only internal subgroup of μ is $\{1\}$. NSS is the key property used in the study of Hilbert's fifth problem. In order to pursue Hilbert's fifth problem in infinite-dimensions, Enflo introduced the following notion:

Definition (Enflo)

G is *UNSS* (uniformly free from small subgroups) if there is a neighborhood U of 1 in G such that, for all neighborhoods V of 1, there is $n_V \in \mathbb{N}$ such that for all $x \in G \setminus V$, $x^n \notin U$ for some $n \leq n_V$.

Lemma (Enflo)

If G is UNSS, then G is locally uniform and metrizable.

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Lemma (Enflo)

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Examples of UNSS groups

- Locally compact groups
- Banach-Lie groups
- **Fact (G.):** If G is a locally exponential Lie group, then G is UNSS if and only if the Lie algebra of G is normable.
- Continuous inverse algebras ("linear groups")
- (Omori) *Strong ILB Lie groups*; in particular, when M is a compact manifold, then $G := \text{Diff}(M)$ is a strong ILB Lie group (that is not locally exponential)

Nonstandard characterization of UNSS

Theorem (G.)

Let G be a locally uniform group. Then the following are equivalent:

1. *G is uniformly NSS;*
2. *G is metrizable and the metric nonstandard hull is uniformly NSS;*
3. *G is metrizable and the metric nonstandard hull is NSS;*
4. *G is metrizable and every nonstandard hull of G is NSS.*

An Example: Unit groups of Banach algebras

- Suppose that $\mathcal{A}$ is a Banach algebra and $G := A^{\times}$, the unit group of $\mathcal{A}$.
- Then G is a Banach-Lie group, whence locally uniform.
- Let d be a left-invariant metric on G and let $\epsilon > 0$ be such that, setting $W := B_d(1;\epsilon)$, we have $W \subseteq \{x \in \mathcal{A} \ : \ \|x - 1\| < 1\} \subseteq G$ and G is W-uniform.
- Let $\hat{\mathcal{A}}$ be the nonstandard hull of $\mathcal{A}$, once again a Banach algebra.

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$\hat{W}$ is a restriction of $U(\hat{\mathcal{A}})$ to a symmetric neighborhood of the origin.

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Pestov's Nonstandard Hull for Banach-Lie groups

- Suppose that G is a Banach-Lie group with Lie algebra $\mathfrak{g}$.
- Let $\exp : \mathfrak{g} \to G$ be the usual exponential map.
- Let $G_{\text{fin},Pestov} := \exp(\mathfrak{g}_{\text{fin}})$
- **Fact:** (Pestov) $G_{\text{fin},Pestov}$ is a group and μ_G is a normal subgroup of G_{fin}. (Uses some nontrivial Lie theory)
- Let $\hat{G}_{Pestov} := G_{\text{fin},Pestov}/\mu_G$.

Theorem (G.)

Suppose that G is a Banach-Lie group and U is such that G is U-uniform. Then $\hat{U}$ is locally isomorphic to $\hat{G}_{Pestov}$.

Theorem (G.)

For suitable U, the Mal'cev hull of $\hat{U}$ is the universal covering group of $\hat{G}_{Pestov}$.

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Another approach

Definition (Henson, Moore)

Suppose that $(X, \mathcal{U})$ is a uniform space. Then $a \in X^*$ is $\mathcal{U}$-finite if, for every $A \in \mathcal{U}$, there is a sequence $a_0, \ldots, a_n \in X^*$ such that $a_0 = a$, $a_n \in X$, and $(a_i, a_{i+1}) \in A^*$ for each $i < n$. Let $X^{\mathcal{U}}_{\text{fin}}$ denote the $\mathcal{U}$-finite points.

It is easy to see that $X^{\mathcal{U}}_{\text{fin}} \subseteq X_{\text{fin},P}$ for any P.

Lemma (G.)

Suppose that G is U-locally uniform and that $V \subseteq U$ is a symmetric neighborhood of 1 satisfying $V^2 \subseteq U$. Then $V^{\mathcal{U}}_{\text{fin}} \cdot V^{\mathcal{U}}_{\text{fin}} \subseteq U^{\mathcal{U}}_{\text{fin}}$ and $(U^{\mathcal{U}}_{\text{fin}})^{-1} = U^{\mathcal{U}}_{\text{fin}}$.

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Another approach (cont'd)

Corollary

If G is a uniform group, then $G^{\mathcal{U}}_{\text{fin}}$ is a subgroup of G_{fin}, yielding a smaller nonstandard hull $\hat{G}^{\mathcal{U}} := G^{\mathcal{U}}_{\text{fin}}/\mu_G \leq \hat{G}_P$.

Example

If E is a locally convex vector space, then $E_{\text{fin}} = E^{\mathcal{U}}_{\text{fin}} = E^{\text{tvs}}_{\text{fin}}$.

Question

Is there a uniform group G such that $\hat{G}^{\mathcal{U}} < \hat{G}_P$?

Question

$X^{\mathcal{U}}_{\text{fin}} = X^*$ if and only if every uniformly continuous function $X \to \mathbb{R}$ is bounded. (Henson) Which groups have this property? (If G satisfies this property and is complete, then every morphism $G \to \mathbb{R}$ is trivial.)

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