TWO SMALL NONSTANDARD GROUP-THEORETIC OBSERVATIONS

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In this note, we make two nonstandard group-theoretic observations. First, we give a short nonstandard proof of a well-known result in group theory concerning locally nilpotent groups. Second, we outline a nonstandard way of viewing the projective limit of an projective family of groups. (We are unsure if this second observation, in some form, has already been discovered, although we would not be surprised to find out that it had.)

1. Locally nilpotent groups

Fix a group G. Recall that G is *locally nilpotent* if every finitely generated subgroup of G is nilpotent. For a nilpotent group H, let $\mathrm{cl}(H)$ denote the nilpotency class of H. Finally, if G is locally nilpotent, set $\mathrm{d}(G) := \sup\{\mathrm{cl}(H) \mid H \leq G, H \text{ finitely generated}\} \in \mathbb{N} \cup \{\infty\}$. We then have the following well-known result.

Theorem 1.1. *If $d(G) \in \mathbb{N}$, then G is nilpotent and* $\mathrm{cl}(G) = \mathrm{d}(G)$.

Proof. The main idea is a well-known method in nonstandard analysis, namely that of *hyperfinite approximation*. Indeed, we embed G into a *hyperfinitely generated* subgroup H of G^*. More precisely, we let $\mathcal{P}_{fg}(G)$ denote the set of finitely generated subgroups of G. Then for each $x \in G$, we let $A_x := \{H \in \mathcal{P}_{fg}(G)^* \mid x \in H\}$. Since the internally cyclic subgroup of G^* generated by x is in A_x, we have that A_x is a nonempty internal set. Moreover, given any finite subset $\{x_1, \ldots, x_n\}$ of G, we have that the internal subgroup of G^* generated by $x_1, \ldots, x_n$ is in $A_{x_1} \cap \cdots \cap A_{x_n}$. Thus, by saturation, there is $H \in \bigcap_x A_x$. Then H is a hyperfinitely generated internal subgroup of G^* containing G as a subgroup. By transfer, we have that H is internally nilpotent. However, since $\mathrm{d}(G) \in \mathbb{N}$, we have that H is actually nilpotent and $\mathrm{cl}(H) \leq \mathrm{d}(G)$. Thus, G is nilpotent and $\mathrm{cl}(G) \leq \mathrm{cl}(H) \leq \mathrm{d}(G)$. But since there is a subgroup of G of nilpotency class $\mathrm{d}(G)$, we must have that $\mathrm{cl}(G) = \mathrm{d}(G)$. □

We should remark that if $\mathrm{d}(G) = \infty$, then the above proof does not work as we cannot pass from the fact that H is internally nilpotent to the fact that H is actually nilpotent. In fact, there are examples of locally nilpotent groups G which are not nilpotent. (Such groups must then necessarily satisfy $\mathrm{d}(G) = \infty$.) An example of such a group is the so-called *generalized dihedral group of* $\mathbb{Z}(2^\infty)$, i.e. $\mathbb{Z}(2^\infty) \rtimes_\phi \mathbb{Z}_2$, where $\phi(0)$ is the identity on $\mathbb{Z}_2$ and $\phi(1)$

is inversion on $\mathbb{Z}_2$. (Here, $\mathbb{Z}(2^\infty) := \{z \in \mathbb{C} \mid z^{2^n} = 1 \text{ for some } n\}$ is the *Prüfer 2-group.*) Indeed, setting $G := \mathbb{Z}(2^\infty) \rtimes_\phi \mathbb{Z}_2$ and identifying $\mathbb{Z}(2^\infty)$ with $\mathbb{Z}(2^\infty) \times \{0\} \leq G$, we see that $[G, G] = \mathbb{Z}(2^\infty) = [\mathbb{Z}(2^\infty), G]$, whence the lower central series for G stabilizes at $\mathbb{Z}(2^\infty)$, implying that G is not nilpotent. However, fix $(x_1, s_1), \ldots, (x_n, s_n) \in G$. Let H_1 be the subgroup of $\mathbb{Z}(2^\infty)$ generated by $x_1, \ldots, x_n$. A key property of $\mathbb{Z}(2^\infty)$ is that every finite subset of it generates a finite 2-group. Thus, $H_1 \rtimes_\phi \mathbb{Z}_2$ is also a finite 2-group, and hence nilpotent. Since the subgroup H of G generated by $(x_1, s_1), \ldots, (x_n, s_n)$ is a subgroup of $H_1 \rtimes_\phi \mathbb{Z}_2$, we see that H is nilpotent. Since H is an arbitrary finitely generated subgroup of G, we see that G is locally nilpotent.

2. Projective limits of groups

Suppose that $(I, \leq)$ is a directed set and (G_i, f_i^j) is a projective family of groups indexed by I, that is, for each $i \in I$, G_i is a group, and for each $i, j \in I$ with $i \leq j$, $f_i^j : G_j \to G_i$ is a homomorphism. The *projective limit of the family* is the group

$$G := \varprojlim G_i := \{(x_i) \in \prod_{i \in I} G_i \ : \ (\forall i \in I)(\forall j \geq i) f_i^j(x_j) = x_i\}.$$

The projective limit satisfies an appropriate universal mapping property and it is the unique group (up to a unique isomorphism) satisfying this property. We now explain how to give a nice nonstandard characterization of this projective limit. To avoid trivialities, we assume that I has no maximal (equivalently, maximum) element.

We first describe how to view the above situation in order for the nonstandard framework to apply smoothly. Let H be a group such that G_i is a subgroup of H for each $i \in I$. Let $\mathrm{Sbgrp}(H)$ denote the set of all subgroups of H and let $\Phi : I \to \mathrm{Sbgrp}(H)$ be defined by $\Phi(i) := G_i$. We consider the nonstandard extension $\Phi : I^* \to \mathrm{Sbgrp}(H)^*$. We can, in the usual way, identify $\mathrm{Sbgrp}(H)^*$ as the set of internal subgroups of H^*, so in this way, for each $i \in I^*$, $\Phi(i)$ is an internal subgroup of H^*. For $i \in I^* \setminus I$, we set $G_i := \Phi(i)$. Also, to encode the homomorphisms f_i^j, we consider the partial function $\Psi : I \times I \times H \rightharpoonup H$ with $\mathrm{dom}(\Psi) := \{(i, j, x) \ : \ i \leq j \text{ and } x \in \Phi(j)\}$ and then for $(i, j, x) \in \mathrm{dom}(\Psi)$, we define $\Psi(i, j, x) := f_i^j(x)$. Again, we have the nonstandard extension $\Psi : I^* \times I^* \times H^* \rightharpoonup H^*$. For any $(i, j) \in I^* \times I^*$ and $x \in \Phi(j)$, we write $f_i^j(x)$ for $\Psi(i, j, x)$; observe that this notation does not clash with the original notation in the situation that $i, j \in I$ and $x \in G_j$.

We say that $i \in I^*$ is *infinite* if $j < i$ for all $j \in I$; we let I_{inf} denote the set of infinite elements of I^*. Observe that $I_{\mathrm{inf}} \neq \emptyset$. To see this, for $j \in I$, let $X_j := \{i \in I^* \mid j < i\}$. Observe that X_j is an internal set that is nonempty by our assumption that I has no maximal element. It follows from the directedness of I that the family $(X_j \ : \ j \in I)$ has the finite intersection property. By saturation, there is $i \in \bigcap_{j \in J} X_j$; such an i is infinite.

For $i \in I_{\text{inf}}$, let $K_i := \{x \in G_i \ : \ (\exists j \in I_{\text{inf}})(j \leq i \text{ and } f^i_j(x) = 1)\}$ and define $\rho_i : G_i \to G$ by $\rho_i(x) = (f^i_j(x))_{j\in I}$. (The fact that $\rho(G_i)$ is a subset of G is a consequence of transfer.)

Proposition 2.1. *For every $i \in I_{\text{inf}}$, ρ_i is a surjective homomorphism and* $\text{Ker}(\rho_i) = K_i$. *Consequently, $G_i/K_i \cong G$.*

Proof. It is clear that ρ_i is a group homomorphism. We first prove that ρ_i is surjective. Fix $g = (g_j)_{j\in I} \in G$. Considering $g : I^* \to H^*$, we claim that $\rho_i(g_i) = g$. Indeed, $(\rho_i(g_i))_j = f^i_j(g_i) = g_j$ by transfer.

Now suppose that $x \in K_i$. Then $f^i_j(x) = 1$ for some $j \in I_{\text{inf}}$ with $j \leq i$. Now fix $k \in I$. Then $(\rho_i(x))_k = f^i_k(x) = f^j_k(f^i_j(x)) = f^j_k(1) = 1$. Since $k \in I$ was arbitrary, we see that $x \in \text{Ker}(\rho_i)$.

Conversely, suppose that $x \in \text{Ker}(\rho_i)$. For $k \in I$, we set

$$Y_k := \{j \in I^* \ : \ k \leq j \text{ and } f^i_j(x) = 1\}.$$

Since I is directed, we have that $(Y_k \ : \ k \in I)$ has the finite intersection property. By saturation, there is $j \in \bigcap_{k\in K} Y_k$; then $j \in Y_{\text{inf}}$ and $f^i_j(x) = 1$, whence $x \in K_i$. □

Note that the proof of the proposition shows that $\rho_i^{-1}((g_j)_{j\in I}) = g_i K_i$. We also know that, for all $i, j \in I_{\text{inf}}$, $G_i/K_i \cong G_j/K_j$. We can be more explicit about the isomorphism witnessing this. From now on, suppose that f^j_i is surjective for each $i, j \in I$ with $i \leq j$. Note then, by transfer, that f^j_i is surjective for each $i, j \in I^*$ with $i \leq j$.

Lemma 2.2. *Suppose that $i, j \in I_{\text{inf}}$ and $i \leq j$. Then the surjective group morphism $f^j_i : G_j \to G_i/K_i$ has kernel K_j. Consequently, the map*

$$xK_j \to f^j_i(x)K_i : G_j/K_j \to G_i/K_i$$

is an isomorphism.

Proof. First suppose that $f^j_i(x) \in K_i$. Then there is $k \in I_{\text{inf}}$ with $k \leq i$ such that $f^i_k(f^j_i(x)) = 1$. However, this then implies that $f^j_k(x) = 1$, so $x \in K_j$.

Conversely, suppose that $x \in K_j$. Take $k \in I_{\text{inf}}$ such that $f^j_k(x) = 1$. For $l \in I$, set $Z_l := \{m \in I^* \ : \ m \leq i \text{ and } l \leq m \text{ and } f^j_m(x) = 1\}$. We claim that $(Z_l \ : \ l \in I)$ has the finite intersection property. Indeed, given $l_1, \ldots, l_r$, take $m \in I$ with $l_1, \ldots, l_r \leq m$. Then $m \leq i$ and $f^j_m(x) = f^k_m(f^j_k(x)) = 1$. Thus, by saturation, there is $m \in \bigcap_{l\in I} Z_l$. Then $m \in I_{\text{inf}}$, $m \leq i$, and $f^i_m(f^j_i(x)) = f^j_m(x) = 1$. Consequently, $f^j_i(x) \in K_i$. □

Corollary 2.3. *Suppose that $i, j \in I_{\text{inf}}$. Let $k \in I_{\text{inf}}$ be such that $i, j \leq k$. Given $x \in G_i$, let $g^k_i(x)$ be any element of G_k such that $f^k_i(g^k_i(x)) = x$. Then the map*

$$xK_i \mapsto f^k_j(g^k_i(x))K_j : G_i/K_i \to G_j/K_j$$

is an isomorphism.

We should say that the above goes through with other algebraic objects, e.g. rings, modules, etc... In all likelihood, this can all be phrased in some abstract, categorical way.