DEFINABLE CLOSURE IN RANDOMIZATIONS

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Abstract. The randomization of a complete first order theory T is the complete continuous theory T^R with two sorts, a sort for random elements of models of T, and a sort for events in an underlying probability space. We give necessary and sufficient conditions for an element to be definable over a set of parameters in a model of T^R.

1. Introduction

A randomization of a first order structure $\mathcal{M}$, as introduced by Keisler [Kei1] and formalized as a metric structure by Ben Yaacov and Keisler [BK], is a continuous structure $\mathcal{N}$ with two sorts, a sort for random elements of $\mathcal{M}$, and a sort for events in an underlying atomless probability space. Given a complete first order theory T, the theory T^R of randomizations of models of T forms a complete theory in continuous logic, which is called the randomization of T. In a model $\mathcal{N}$ of T^R, for each n-tuple $\vec{a}$ of random elements and each first order formula $\varphi(\vec{v})$, the set of points in the underlying probability space where $\varphi(\vec{a})$ is true is an event denoted by $[\![\varphi(\vec{a})]\!]$.

In a first order structure $\mathcal{M}$, an element b is *definable over* a set A of elements of $\mathcal{M}$ (called parameters) if there is a tuple $\vec{a}$ in A and a formula $\varphi(u, \vec{a})$ such that

$$\mathcal{M} \models (\forall u)(\varphi(u, \vec{a}) \leftrightarrow u = b).$$

In a general metric structure $\mathcal{N}$, an element b is said to be *definable over* a set of parameters A if there is a sequence of tuples $\vec{a}_n$ in A and continuous formulas $\Phi_n(x, \vec{a}_n)$ whose truth values converge uniformly to the distance from x to b. In this paper we give necessary and sufficient conditions for definability in a model of the randomization theory T^R. These conditions can be stated in terms of sequences of first order formulas. The results in this paper will be applied in a forthcoming paper about independence relations in randomizations.

In Theorem 3.1.2, we show that an event E is definable over a set A of parameters if and only if it is the limit of a sequence of events of the form $[\![\varphi_n(\vec{a}_n)]\!]$, where each φ_n is a first order formula and each $\vec{a}_n$ is a tuple from A.

In Theorem 3.3.6, we show that a random element b is definable over a set A of parameters if and only if b is the limit of a sequence of random elements

2010 *Mathematics Subject Classification.* Primary 03C40. Secondary 03B48, 03B50, 03C35.

b_n such that for each n,

$$[\![(\forall u)(\varphi_n(u,\vec{a}_n) \leftrightarrow u = b_n)]\!]$$

has probability one for some first order formula $\varphi_n(u,\vec{v})$ and a tuple $\vec{a}_n$ from A. In Section 4 we give some consequences in the special case that the underlying first order theory T is $\aleph_0$-categorical.

Continuous model theory in its current form is developed in the papers [BBHU] and [BU]. The papers [Go1], [Go2], [Go3] deal with definability questions in metric structures. Randomizations of models are treated in [AK], [Be], [BK], [EG], [GL], [Ke1], and [Ke2].

2. Preliminaries

We refer to [BBHU] and [BU] for background in continuous model theory, and follow the notation of [BK]. We assume familiarity with the basic notions about continuous model theory as developed in [BBHU], including the notions of a theory, structure, pre-structure, model of a theory, elementary extension, isomorphism, and κ-saturated structure. In particular, the universe of a pre-structure is a pseudo-metric space, the universe of a structure is a complete metric space, and every pre-structure has a unique completion. In continuous logic, formulas have truth values in the unit interval $[0,1]$ with 0 meaning true, the connectives are continuous functions from $[0,1]^n$ into $[0,1]$, and the quantifiers are sup and inf. A *tuple* is a finite sequence, and $A^{<\mathbb{N}}$ is the set of all tuples of elements of A.

2.1. The theory T^R.

We assume throughout that L is a finite or countable first order signature, and that T is a complete theory for L whose models have at least two elements.

The *randomization signature* L^R is the two-sorted continuous signature with sorts $\mathbb{K}$ (for random elements) and $\mathbb{B}$ (for events), an n-ary function symbol $[\![\varphi(\cdot)]\!]$ of sort $\mathbb{K}^n \to \mathbb{B}$ for each first order formula φ of L with n free variables, a $[0,1]$-valued unary predicate symbol μ of sort $\mathbb{B}$ for probability, and the Boolean operations $\top, \bot, \sqcap, \sqcup, \neg$ of sort $\mathbb{B}$. The signature L^R also has distance predicates $d_{\mathbb{B}}$ of sort $\mathbb{B}$ and $d_{\mathbb{K}}$ of sort $\mathbb{K}$. In L^R, we use $\mathsf{B}, \mathsf{C}, \ldots$ for variables or parameters of sort $\mathbb{B}$. $\mathsf{B} \doteq \mathsf{C}$ means $d_{\mathbb{B}}(\mathsf{B},\mathsf{C}) = 0$, and $\mathsf{B} \sqsubseteq \mathsf{C}$ means $\mathsf{B} \doteq \mathsf{B} \sqcap \mathsf{C}$.

A pre-structure for T^R will be a pair $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ where $\mathcal{K}$ is the part of sort $\mathbb{K}$ and $\mathcal{B}$ is the part of sort $\mathbb{B}$. The *reduction* of $\mathcal{P}$ is the pre-structure $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ obtained from $\mathcal{P}$ by identifying elements at distance zero, and the associated mapping from $\mathcal{P}$ onto $\mathcal{N}$ is called the *reduction map*. The *completion* of $\mathcal{P}$ is the structure obtained by completing the metrics in the reduction of $\mathcal{P}$. A pre-structure $\mathcal{P}$ is called *pre-complete* if the reduction of $\mathcal{P}$ is already the completion of $\mathcal{P}$.

In [BK], the randomization theory T^R is defined by listing a set of axioms. We will not repeat these axioms here, because it is simpler to give the following model-theoretic characterization of T^R.

Definition 2.1.1. Given a model $\mathcal{M}$ of T, a *nice randomization of* $\mathcal{M}$ is a pre-complete structure $(\mathcal{K},\mathcal{B})$ for L^R equipped with an atomless probability space $(\Omega,\mathcal{B},\mu)$ such that:

(1) $\mathcal{B}$ is a σ-algebra with $\top,\bot,\sqcap,\sqcup,\neg$ interpreted by $\Omega,\emptyset,\cap,\cup,\setminus$.
(2) $\mathcal{K}$ is a set of functions $a\colon\Omega\to M$.
(3) For each formula $\psi(\vec{x})$ of L and tuple $\vec{a}$ in $\mathcal{K}$, we have
$$\llbracket\psi(\vec{a})\rrbracket=\{\omega\in\Omega:\mathcal{M}\models\psi(\vec{a}(\omega))\}\in\mathcal{B}.$$
(4) $\mathcal{B}$ is equal to the set of all events $\llbracket\psi(\vec{a})\rrbracket$ where $\psi(\vec{v})$ is a formula of L and $\vec{a}$ is a tuple in $\mathcal{K}$.
(5) For each formula $\theta(u,\vec{v})$ of L and tuple $\vec{b}$ in $\mathcal{K}$, there exists $a\in\mathcal{K}$ such that
$$\llbracket\theta(a,\vec{b})\rrbracket=\llbracket(\exists u\,\theta)(\vec{b})\rrbracket.$$
(6) On $\mathcal{K}$, the distance predicate $d_{\mathbb{K}}$ defines the pseudo-metric
$$d_{\mathbb{K}}(a,b)=\mu\llbracket a\neq b\rrbracket.$$
(7) On $\mathcal{B}$, the distance predicate $d_{\mathbb{B}}$ defines the pseudo-metric
$$d_{\mathbb{B}}(\mathsf{B},\mathsf{C})=\mu(\mathsf{B}\triangle\mathsf{C}).$$

Definition 2.1.2. For each first order theory T, the *randomization theory* T^R is the set of sentences that are true in all nice randomizations of models of T.

It follows that for each first order sentence φ, if $T\models\varphi$ then $T^R\models\llbracket\varphi\rrbracket\doteq\top$. The following basic facts are from [BK], Theorem 2.1 and Proposition 2.2, Example 3.4 (ii), Proposition 2.7, and Theorem 2.9.

Fact 2.1.3. For every complete first order theory T, the randomization theory T^R is complete.

Fact 2.1.4. Every model $\mathcal{M}$ of T has nice randomizations.

Fact 2.1.5. (Fullness) Every pre-complete model $\mathcal{P}=(\mathcal{K},\mathcal{B})$ of T^R has perfect witnesses, i.e.,

(1) For each first order formula $\theta(u,\vec{v})$ and each $\vec{b}$ in $\mathcal{K}^n$ there exists $a\in\mathcal{K}$ such that
$$\llbracket\theta(a,\vec{b})\rrbracket\doteq\llbracket(\exists u\,\theta)(\vec{b})\rrbracket;$$
(2) For each $\mathsf{B}\in\mathcal{B}$ there exist $a,b\in\mathcal{K}$ such that $\mathsf{B}\doteq\llbracket a=b\rrbracket$.

Corollary 2.1.6. *Every model $\mathcal{N}$ of T^R has a pair of elements c,d such that $\llbracket c\neq d\rrbracket=\top$.*

Proof. Every model of T has at least two elements, so $T\models(\exists u)(\exists v)u\neq v$. The result follows by applying Fullness twice. ☐

Fact 2.1.7. (Strong quantifier elimination) Every formula Φ in the continuous language L^R is T^R-equivalent to a formula with the same free variables and no quantifiers of sort $\mathbb{K}$ or $\mathbb{B}$.

Lemma 2.1.8. *Let $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ be a pre-complete model of T^R and let $a, b \in \mathcal{K}$ and $\mathsf{B} \in \mathcal{B}$. Then there is an element $c \in \mathcal{K}$ that agrees with a on B and agrees with b on $\neg\mathsf{B}$, that is, $\mathsf{B} \sqsubseteq [\![c = a]\!]$ and $(\neg\mathsf{B}) \sqsubseteq [\![c = b]\!]$.*

Definition 2.1.9. In Lemma 2.1.8, we will call c a *characteristic function of* B *with respect to* a, b.

Note that the distance between any two characteristic functions of an event B with respect to elements a, b is zero. In particular, in a model of T^R, the characteristic function is unique.

Proof of Lemma 2.1.8. By Fact 2.1.5 (2), there exist $d, e \in \mathcal{K}$ such that $\mathsf{B} \doteq [\![d = e]\!]$. The first order sentence

$$(\forall u)(\forall v)(\forall x)(\forall y)(\exists z)[(x = y \rightarrow z = u) \wedge (x \neq y \rightarrow z = v)]$$

is logically valid, so we must have

$$[\![(\exists z)[(d = e \rightarrow z = a) \wedge (d \neq e \rightarrow z = b)]]\!] \doteq \top.$$

By Fact 2.1.5 (1) there exists $c \in \mathcal{K}$ such that

$$[\![d = e \rightarrow c = a]\!] \doteq \top, \quad [\![d \neq e \rightarrow c = b]\!] \doteq \top,$$

so $[\![d = e]\!] \sqsubseteq [\![c = a]\!]$ and $[\![d \neq e]\!] \sqsubseteq [\![c = b]\!]$. $\square$

We will need the following result, which is a consequence of Theorem 3.11 of [Be]. Since the setting in [Be] is quite different from the present paper, we give a direct proof here.

Proposition 2.1.10. *Every model of T^R is isomorphic to the reduction of a nice randomization of a model of T.*

Proof. Let $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ be a model of T^R of cardinality κ. Let Ω be the Stone space of the Boolean algebra $\widehat{\mathcal{B}} = (\widehat{\mathcal{B}}, \top, \bot, \sqcap, \sqcup, \neg)$. Thus Ω is a compact topological space, the points of Ω are ultrafilters, we may identify $\widehat{\mathcal{B}}$ with the Boolean algebra of clopen sets of Ω, and $\mu^{\mathcal{N}}$ is a finitely additive probability measure on $\widehat{\mathcal{B}}$.

We next show that μ is σ-additive on $\widehat{\mathcal{B}}$. To do this, we assume that $\mathsf{A}_0 \supseteq \mathsf{A}_1 \supseteq \cdots$ in $\widehat{\mathcal{B}}$ and $\mathsf{C} = \bigcap_n \mathsf{A}_n \in \widehat{\mathcal{B}}$, and prove that $\mu(\mathsf{C}) = \lim_{n\to\infty} \mu(\mathsf{A}_n)$. Indeed, the family $\{\mathsf{C} \cup (\Omega \setminus \mathsf{A}_n) \colon n \in \mathbb{N}\}$ is an open covering of Ω, so by the topological compactness of Ω, we have $\Omega = \bigcup_{k=0}^{n} (\mathsf{C} \cup (\Omega \setminus \mathsf{A}_k))$ for some $n \in \mathbb{N}$. Then $\mathsf{C} = \mathsf{A}_n$, so $\mu(\mathsf{C}) = \mu(\mathsf{A}_n) = \lim_{n\to\infty} \mu(\mathsf{A}_n)$.

By the Caratheodory theorem, there is a complete probability space $(\Omega, \mathcal{B}, \mu)$ such that $\mathcal{B} \supseteq \widehat{\mathcal{B}}$, μ agrees with $\mu^{\mathcal{N}}$ on $\widehat{\mathcal{B}}$, and for each $\mathsf{B} \in \mathcal{B}$ and $m > 0$ there is a countable sequence $\mathsf{A}_{m0} \subseteq \mathsf{A}_{m1} \subseteq \cdots$ in $\widehat{\mathcal{B}}$ such that

$$B \subseteq \bigcup_n \mathsf{A}_{mn} \text{ and } \mu\left(\bigcup_n \mathsf{A}_{mn}\right) \leq \mu(\mathsf{B}) + 1/m. \tag{2.1}$$

Note that since the probability space $(\Omega, \mathcal{B}, \mu)$ is complete, every subset of Ω that contains a set in $\mathcal{B}$ of measure one also belongs to $\mathcal{B}$ and has measure one.

We claim that for each $\mathsf{B} \in \mathcal{B}$ there is a unique event $f(\mathsf{B}) \in \widehat{\mathcal{B}}$ such that $\mu(f(\mathsf{B}) \triangle \mathsf{B}) = 0$. The uniqueness of $f(\mathsf{B})$ follows from the fact that the distance function $d_{\mathbb{B}}(\mathsf{C}, \mathsf{D}) = \mu(\mathsf{C} \triangle \mathsf{D})$ is a metric on $\widehat{\mathcal{B}}$. To show the existence of $f(\mathsf{B})$, for each $m > 0$ let $\mathsf{A}_{m0} \subseteq \mathsf{A}_{m1} \subseteq \cdots$ be as in (2.1). Note that $(\mathsf{A}_{m0}, \mathsf{A}_{m1}, \ldots)$ is a Cauchy sequence of events in the model $\mathcal{N}$, so there is an event $\mathsf{C}_m \in \widehat{\mathcal{B}}$ such that $\mathsf{C}_m = \lim_{n\to\infty} \mathsf{A}_{mn}$. Hence $\lim_{n\to\infty} \mu(\mathsf{A}_{mn} \triangle \mathsf{C}_m) = 0$, so $\mu((\bigcup_n \mathsf{A}_{mn}) \triangle \mathsf{C}_m) = 0$. Then $(\mathsf{C}_1, \mathsf{C}_2, \ldots)$ is a Cauchy sequence, so there is an event $f(\mathsf{B}) = \lim_{m\to\infty} \mathsf{C}_m$ in $\widehat{\mathcal{B}}$ with $\mu(f(\mathsf{B}) \triangle \mathsf{B}) = 0$.

We make some observations about the mapping $f \colon \mathcal{B} \to \widehat{\mathcal{B}}$. If $\mathsf{B}, \mathsf{C} \in \mathcal{B}$ and $d_{\mathbb{B}}(\mathsf{B}, \mathsf{C}) = 0$, then $f(\mathsf{B}) = f(\mathsf{C})$. For each $\mathsf{B}, \mathsf{C} \in \mathcal{B}$, we have

$$f(\mathsf{B} \cup \mathsf{C}) = f(\mathsf{B}) \cup f(\mathsf{C}), \qquad f(\mathsf{B} \cap \mathsf{C}) = f(\mathsf{B}) \cap f(\mathsf{C}),$$

$$\Omega \setminus f(\mathsf{B}) = f(\Omega \setminus \mathsf{B}), \qquad \mu(\mathsf{B}) = \mu(f(\mathsf{B})).$$

Moreover, the mapping f sends $\mathcal{B}$ onto $\widehat{\mathcal{B}}$, because if $\mathsf{C} \in \widehat{\mathcal{B}}$ then $\mathsf{C} \in \mathcal{B}$ and $f(\mathsf{C}) = \mathsf{C}$. Therefore the mapping $\widehat{f}$ that sends the equivalence class of each $\mathsf{B} \in \mathcal{B}$ under $d_{\mathbb{B}}$ to $f(\mathsf{B})$ is well defined and is an isomorphism from the reduction of the pre-structure $(\mathcal{B}, \sqcup, \sqcap, \neg. \top, \bot, \mu)$ onto the measured algebra $(\widehat{\mathcal{B}}, \sqcup, \sqcap, \neg. \top, \bot, \mu)$.

A model $\mathcal{M}$ of T is κ^+*-universal* if every model of T of cardinality $\le \kappa$ is elementarily embeddable in $\mathcal{M}$. By Theorem 5.1.12 in [CK], every κ-saturated model of T is κ^+-universal, so κ^+-universal models of T exist. We now assume that $\mathcal{M}$ is a κ^+-universal model of T, and prove that $\mathcal{N}$ is isomorphic to the reduction of a nice randomization of $\mathcal{M}$ with the underlying probability space $(\Omega, \mathcal{B}, \mu)$.

In the following paragraphs, we will use boldface letters $\boldsymbol{b}, \boldsymbol{d}, \ldots$ for elements of $\widehat{\mathcal{K}}$. Let $L_{\widehat{\mathcal{K}}}$ be the first order signature formed by adding a constant symbol for each element $\boldsymbol{b} \in \widehat{\mathcal{K}}$. For each $\omega \in \Omega$, the set of $L_{\widehat{\mathcal{K}}}$-sentences

$$U(\omega) = \{\psi(\vec{\boldsymbol{b}}) \colon \omega \in [\![\psi(\vec{\boldsymbol{b}})]\!]\}$$

is consistent with T and has cardinality $\le \kappa$. By the Compactness and Löwenheim-Skolem theorems, each $U(\omega)$ has a model $(\mathcal{M}_\omega, \boldsymbol{b}_\omega)_{\boldsymbol{b} \in \widehat{\mathcal{K}}}$ of cardinality $\le \kappa$. Since $\mathcal{M}$ is κ^+-universal, for each $\omega \in \Omega$ we may choose an elementary embedding $h_\omega \colon \mathcal{M}_\omega \prec \mathcal{M}$. Then $(\mathcal{M}, h_\omega(\boldsymbol{b}_\omega))_{\boldsymbol{b} \in \widehat{\mathcal{K}}} \models U(\omega)$ for every $\omega \in \Omega$. It follows that for each formula $\psi(\vec{v})$ of L and each tuple $\vec{\boldsymbol{b}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$,

$$[\![\psi(\vec{\boldsymbol{b}})]\!] = \{\omega \in \Omega \colon \mathcal{M}_\omega \models \psi(\vec{\boldsymbol{b}}_\omega)\} = \{\omega \in \Omega \colon \mathcal{M} \models \psi(h_\omega(\vec{\boldsymbol{b}}_\omega))\} \in \widehat{\mathcal{B}}.$$

For each formula $\psi(\vec{v})$ of L and tuple $\vec{c}$ of functions in M^Ω, define

$$[\![\psi(\vec{c})]\!] := \{\omega \in \Omega \colon \mathcal{M} \models \psi(\vec{c}(\omega))\}.$$

Let $\mathcal{K}$ be the set of all functions $a\colon \Omega \to M$ such that for some element $\boldsymbol{b} \in \widehat{\mathcal{K}}$, we have

$$\mu(\{\omega \in \Omega \colon a(\omega) = h_\omega(\boldsymbol{b}_\omega)\}) = 1.$$

We claim that for each $a \in \mathcal{K}$ there is a unique element $f(a) \in \widehat{\mathcal{K}}$ such that

$$\mu(\{\omega \in \Omega \colon a(\omega) = h_\omega(f(a)_\omega)\}) = 1.$$

The existence of $f(a)$ is guaranteed by the definition of $\mathcal{K}$. To prove uniqueness, suppose $\boldsymbol{b}, \boldsymbol{d} \in \widehat{\mathcal{K}}$ and

$$\mu(\{\omega \in \Omega \colon a(\omega) = h_\omega(\boldsymbol{b}_\omega)\}) = \mu(\{\omega \in \Omega \colon a(\omega) = h_\omega(\boldsymbol{d}_\omega)\}) = 1.$$

Then

$$\mu(\{\omega \in \Omega \colon h_\omega(\boldsymbol{b}_\omega) = h_\omega(\boldsymbol{d}_\omega)\}) = 1,$$

so

$$\mu(\llbracket \boldsymbol{b} = \boldsymbol{d} \rrbracket) = \mu(\{\omega \in \Omega \colon \boldsymbol{b}_\omega = \boldsymbol{d}_\omega\}) = 1,$$

and hence $d_{\mathbb{K}}(\boldsymbol{b}, \boldsymbol{d}) = 0$. Since $d_{\mathbb{K}}$ is a metric on $\widehat{\mathcal{K}}$, it follows that $\boldsymbol{b} = \boldsymbol{d}$.

We now make some observations about the mapping $f\colon \mathcal{K} \to \widehat{\mathcal{K}}$. This mapping sends $\mathcal{K}$ onto $\widehat{\mathcal{K}}$, because for each $\boldsymbol{b} \in \widehat{\mathcal{K}}$, we have $f(a) = \boldsymbol{b}$ where a is the element of $\mathcal{K}$ such that $a(\omega) = h_\omega(\boldsymbol{b}_\omega)$ for all $\omega \in \Omega$. Suppose $\vec{c} \in \mathcal{K}^{<\mathbb{N}}$ and $\vec{\boldsymbol{d}} = f(\vec{c})$. We have $\vec{\boldsymbol{d}} \in \widehat{\mathcal{K}}^{<\mathbb{N}}$ and

$$\llbracket \psi(\vec{\boldsymbol{d}}) \rrbracket = \{\omega \in \Omega \colon \mathcal{M} \models \psi(h_\omega(\vec{\boldsymbol{d}}_\omega))\} \doteq \{\omega \in \Omega \colon \mathcal{M} \models \psi(\vec{c}(\omega))\} = \llbracket \psi(\vec{c}) \rrbracket.$$

Since the probability space $(\Omega, \mathcal{B}, \mu)$ is complete, $\llbracket \psi(\vec{\boldsymbol{d}}) \rrbracket \in \widehat{\mathcal{B}} \subseteq \mathcal{B}$, and $\llbracket \psi(\vec{\boldsymbol{d}}) \rrbracket \doteq \llbracket \psi(\vec{c}) \rrbracket$, we have $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$ and $\llbracket \psi(\vec{\boldsymbol{d}}) \rrbracket = f(\llbracket \psi(\vec{c}) \rrbracket)$. Therefore, if $a, c \in \mathcal{K}$ and $d_{\mathbb{K}}(a, c) = 0$, then $d_{\mathbb{K}}(f(a), f(c)) = 0$, and hence $f(a) = f(c)$. This shows that $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ is a well-defined pre-complete structure for L^R, and that the mapping $\widehat{f}$ that sends the equivalence class of each $\mathsf{B} \in \mathcal{B}$ to $f(\mathsf{B})$, and the equivalence class of each $a \in \mathcal{K}$ to $f(a)$, is an isomorphism from the reduction of $\mathcal{P}$ to $\mathcal{N}$.

It remains to show that $\mathcal{P}$ is a nice randomization of $\mathcal{M}$. It is clear that $\mathcal{P}$ satisfies conditions (1)-(3) in Definition 2.1.1.

Proof of (4): We have already shown that $\llbracket \psi(\vec{c}) \rrbracket \in \mathcal{B}$ for each formula $\psi(\vec{v})$ of L and each tuple $\vec{c}$ in $\mathcal{K}$. For the other direction, let $\mathsf{B} \in \mathcal{B}$. By Corollary 2.1.6, there exist $a, e \in \mathcal{K}$ such that $\llbracket a \neq e \rrbracket \doteq \Omega$. We may choose a function $b \in M^\Omega$ such that $b(\omega) = e(\omega)$ whenever $a(\omega) \neq e(\omega)$, and $b(\omega) \neq a(\omega)$ for all $\omega \in \Omega$. Then $b \in \mathcal{K}$ and $\llbracket a \neq b \rrbracket = \Omega$. By Lemma 2.1.8, there exists $c \in \mathcal{K}$ which is a characteristic function of B with respect to a, b. Then $\llbracket c = a \rrbracket \doteq \mathsf{B}$. Let $d \in M^\Omega$ be the function such that $d(\omega) = a(\omega)$ for $\omega \in \mathsf{B}$, and $d(\omega) = b(\omega)$ for $\omega \in \neg\mathsf{B}$. Then $\mu(\llbracket c = d \rrbracket) = 1$, so $d \in \mathcal{K}$, and $\llbracket a = d \rrbracket = \mathsf{B}$. Thus (4) holds with ψ being the sentence $a = d$.

Proof of (5): Consider a formula $\theta(u, \vec{v})$ of L and a tuple $\vec{b}$ in $\mathcal{K}$. By Fullness, there exists $c \in \mathcal{K}$ such that

$$\llbracket \theta(c, \vec{b}) \rrbracket \doteq \llbracket (\exists u)\theta(u, \vec{b}) \rrbracket.$$

We may choose a function $a \in M^\Omega$ such that for all $\omega \in \Omega$,

$$\mathcal{M} \models [\theta(c(\omega), \vec{b}(\omega)) \leftrightarrow (\exists u)\theta(u, \vec{b})] \text{ implies } a(\omega) = c(\omega),$$

and

$$\mathcal{M} \models [(\exists u)\theta(u, \vec{b}(\omega)) \to \theta(a(\omega), \vec{b}(\omega))].$$

Then $\mu([\![a = c]\!]) = 1$, so $a \in \mathcal{K}$ and

$$[\![\theta(a, \vec{b})]\!] = [\![(\exists u)\theta(u, \vec{b})]\!],$$

as required.

Proof of (6) and (7): By Fact 2.1.4, the properties

$$(\forall x)(\forall y)d_{\mathbb{K}}(x, y) = \mu([\![x \neq y]\!]), \quad (\forall \mathsf{U})(\forall \mathsf{V})d_{\mathbb{B}}(\mathsf{U}, \mathsf{V}) = \mu(\mathsf{U} \triangle \mathsf{V})$$

hold in some model of T^R. By Fact 2.1.3, these properties hold in all models of T^R, and thus in $\mathcal{N}$. Therefore (6) and (7) hold for $\mathcal{P}$. $\square$

2.2. **Types and Definability.** For a first order structure $\mathcal{M}$ and a set A of elements of $\mathcal{M}$, $\mathcal{M}_A$ denotes the structure formed by adding a new constant symbol to $\mathcal{M}$ for each $a \in A$. The *type realized by* a tuple $\vec{b}$ over the parameter set A in $\mathcal{M}$ is the set $\mathrm{tp}^{\mathcal{M}}(\vec{b}/A)$ of formulas $\varphi(\vec{u}, \vec{a})$ with $\vec{a} \in A^{<\mathbb{N}}$ satisfied by $\vec{b}$ in $\mathcal{M}_A$. We call $\mathrm{tp}^{\mathcal{M}}(\vec{b}/A)$ an *n-type* if $n = |\vec{b}|$.

In the following, let $\mathcal{N}$ be a continuous structure and let A be a set of elements of $\mathcal{N}$. $\mathcal{N}_A$ denotes the structure formed by adding a new constant symbol to $\mathcal{N}$ for each $a \in A$. The *type* $\mathrm{tp}^{\mathcal{N}}(\vec{b}/A)$ *realized* by $\vec{b}$ over the parameter set A in $\mathcal{N}$ is the function p from formulas to $[0, 1]$ such that for each formula $\Phi(\vec{x}, \vec{a})$ with $\vec{a} \in A^{<\mathbb{N}}$, we have $\Phi(\vec{x}, \vec{a})^p = \Phi(\vec{b}, \vec{a})^{\mathcal{N}}$.

We now recall the notions of definable element and algebraic element from [BBHU]. An element b is *definable over* A in $\mathcal{N}$, in symbols $b \in \mathrm{dcl}^{\mathcal{N}}(A)$, if there is a sequence of formulas $\langle \Phi_k(x, \vec{a}_k)\rangle$ with $\vec{a}_k \in A^{<\mathbb{N}}$ such that the sequence of functions $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}}\rangle$ converges uniformly in x to the distance function $d(x, b)^{\mathcal{N}}$ of the corresponding sort. b is *algebraic over* A in $\mathcal{N}$, in symbols $b \in \mathrm{acl}^{\mathcal{N}}(A)$, if there is a compact set C and a sequence of formulas $\langle \Phi_k(x, \vec{a}_k)\rangle$ with $\vec{a}_k \in A^{<\mathbb{N}}$ such that $b \in C$ and the sequence of functions $\langle \Phi_k(x, \vec{a}_k)^{\mathcal{N}}\rangle$ converges uniformly in x to the distance function $d(x, C)^{\mathcal{N}}$ of the corresponding sort.

If the structure $\mathcal{N}$ is clear from the context, we will sometimes drop the superscript and write $\mathrm{tp}, \mathrm{dcl}, \mathrm{acl}$ instead of $\mathrm{tp}^{\mathcal{N}}, \mathrm{dcl}^{\mathcal{N}}, \mathrm{acl}^{\mathcal{N}}$.

Fact 2.2.1. ([BBHU], Exercises 10.7 and 10.10) For each element b of $\mathcal{N}$, the following are equivalent, where $p = \mathrm{tp}^{\mathcal{N}}(b/A)$:

(1) b is definable over A in $\mathcal{N}$;
(2) in each model $\mathcal{N}' \succ \mathcal{N}$, b is the a unique element that realizes p over A;
(3) b is definable over some countable subset of A in $\mathcal{N}$.

Fact 2.2.2. ([BBHU], Exercise 10.8 and 10.11) For each element b of $\mathcal{N}$, the following are equivalent, where $p = \mathrm{tp}^{\mathcal{N}}(b/A)$:

(1) b is algebraic over A in $\mathcal{N}$;
(2) in each model $\mathcal{N}' \succ \mathcal{N}$, the set of elements b that realize p over A in $\mathcal{N}'$ is compact.
(3) b is algebraic over some countable subset of A in $\mathcal{N}$.

Fact 2.2.3. (Definable Closure, Exercises 10.10 and 10.11 in [BBHU])
(1) If $A \subseteq \mathcal{N}$ then $\mathrm{dcl}(A) = \mathrm{dcl}(\mathrm{dcl}(A))$ and $\mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$.
(2) If A is a dense subset of B and $B \subseteq \mathcal{N}$, then $\mathrm{dcl}(A) = \mathrm{dcl}(B)$ and $\mathrm{acl}(A) = \mathrm{acl}(B)$.

It follows that for any $A \subseteq \mathcal{N}$, $\mathrm{dcl}(A)$ and $\mathrm{acl}(A)$ are closed with respect to the metric in $\mathcal{N}$.

We now turn to the case where $\mathcal{N}$ is a model of T^R. In that case, a set of elements of $\mathcal{N}$ may contain elements of both sorts $\mathbb{K}, \mathbb{B}$. But as we will now explain, we need only consider definability over sets of parameters of sort $\mathbb{K}$.

Remark 2.2.4. Let $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ be a model of T^R. Since every model of T has at least two elements, $\mathcal{N}$ has a pair of elements a, b of sort $\mathbb{K}$ such that $\mathcal{N} \models [\![a = b]\!] = \bot$. For each event $\mathsf{D} \in \widehat{\mathcal{B}}$, let 1_{D} be the characteristic function of D with respect to a, b. Then in the model $\mathcal{N}$, D is definable over $\{a, b, 1_{\mathsf{D}}\}$, and 1_{D} is definable over $\{a, b, \mathsf{D}\}$.

Proof. By Fact 2.2.1. □

In view of Remark 2.2.4 and Fact 2.2.3, if C is a set of parameters in $\mathcal{N}$ of both sorts, and there are elements $a, b \in C$ such that $\mathcal{N} \models [\![a = b]\!] = \bot$, then an element of either sort is definable over C if and only if it is definable over the set of parameters of sort $\mathbb{K}$ obtained by replacing each element of C of sort $\mathbb{B}$ by its characteristic function with respect to a, b. For this reason, in a model $\mathcal{N}$ of T^R we will only consider definability over sets of parameters of sort $\mathbb{K}$. We write $\mathrm{dcl}_{\mathbb{B}}(A)$ for the set of elements of sort $\mathbb{B}$ that are definable over A in $\mathcal{N}$, and write $\mathrm{dcl}(A)$ for the set of elements of sort $\mathbb{K}$ that are definable over A in $\mathcal{N}$. Similarly for $\mathrm{acl}_{\mathbb{B}}(A)$ and $\mathrm{acl}(A)$.

2.3. **Conventions and Notation.** We will assume hereafter that $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ is a model of T^R, $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ is a nice randomization of a model $\mathcal{M} \models T$ with probability space $(\Omega, \mathcal{B}, \mu)$, and $\mathcal{N}$ is the reduction of $\mathcal{P}$. The existence of $\mathcal{P}$ is guaranteed by Proposition 2.1.10.

We will use boldfaced letters $\boldsymbol{a}, \boldsymbol{b}, \ldots$ for elements of $\widehat{\mathcal{K}}$. For each element $\boldsymbol{a} \in \widehat{\mathcal{K}}$, we will choose once and for all an element $a \in \mathcal{K}$ such that the image of a under the reduction map is $\boldsymbol{a}$. It follows that for each first order formula $\varphi(\vec{v})$, $[\![\varphi(\vec{\boldsymbol{a}})]\!]$ is the image of $[\![\varphi(\vec{a})]\!]$ under the reduction map. For any countable set $A \subseteq \widehat{\mathcal{K}}$ and each $\omega \in \Omega$, we define

$$A(\omega) = \{a(\omega) \colon \boldsymbol{a} \in A\}.$$

When $A \subseteq \widehat{\mathcal{K}}$, $\mathrm{cl}(A)$ denotes the closure of A in the metric $d_{\mathbb{K}}$. When $B \subseteq \widehat{\mathcal{B}}$, $\mathrm{cl}(B)$ denotes the closure of B in the metric $d_{\mathbb{B}}$, and $\sigma(B)$ denotes the smallest σ-subalgebra of $\widehat{\mathcal{B}}$ containing B.

3. Randomizations of Arbitrary Theories

3.1. **Definability in Sort $\mathbb{B}$.** We characterize the set of elements of $\widehat{\mathcal{B}}$ that are definable in $\mathcal{N}$ over a set of parameters $A \subseteq \widehat{\mathcal{K}}$.

Definition 3.1.1. For each $A \subseteq \widehat{\mathcal{K}}$, we say that an event E is *first order definable* over A, in symbols $\mathsf{E} \in \operatorname{fdcl}_{\mathbb{B}}(A)$, if $\mathsf{E} = [\![\varphi(\vec{\boldsymbol{a}})]\!]$ for some first order formula $\varphi(\vec{v})$ and tuple $\vec{\boldsymbol{a}}$ in $A^{<\mathbb{N}}$.

Theorem 3.1.2. *For each $A \subseteq \widehat{\mathcal{K}}$, $\operatorname{dcl}_{\mathbb{B}}(A) = \operatorname{cl}(\operatorname{fdcl}_{\mathbb{B}}(A)) = \sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$.*

Proof. By quantifier elimination (Fact 2.1.7), in any elementary extension $\mathcal{N}' \succ \mathcal{N}$, two events have the same type over A if and only if they have the same type over $\operatorname{fdcl}_{\mathbb{B}}(A)$. Then by Fact 2.2.1, $\operatorname{dcl}_{\mathbb{B}}(A) = \operatorname{dcl}_{\mathbb{B}}(\operatorname{fdcl}_{\mathbb{B}}(A))$. Moreover, $\operatorname{dcl}_{\mathbb{B}}(\operatorname{fdcl}_{\mathbb{B}}(A))$ is equal to the definable closure of $\operatorname{fdcl}_{\mathbb{B}}(A)$ in the pure measured algebra $(\widehat{\mathcal{B}}, \mu)$. By Observation 16.7 in [BBHU], the definable closure of $\operatorname{fdcl}_{\mathbb{B}}(A)$ in $(\widehat{\mathcal{B}}, \mu)$ is equal to $\sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$, so $\operatorname{dcl}_{\mathbb{B}}(A) = \sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$. Since $\operatorname{fdcl}_{\mathbb{B}}(A)$ is a Boolean subalgebra of $\widehat{\mathcal{B}}$, $\operatorname{cl}(\operatorname{fdcl}_{\mathbb{B}}(A))$ is a Boolean subalgebra of $\widehat{\mathcal{B}}$. By metric completeness, $\operatorname{cl}(\operatorname{fdcl}_{\mathbb{B}}(A))$ is a σ-algebra and $\sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$ is closed, so $\operatorname{cl}(\operatorname{fdcl}_{\mathbb{B}}(A)) = \sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$. □

Corollary 3.1.3. *The only events that are definable without parameters in $\mathcal{N}$ are $\top$ and $\bot$.*

Proof. For every first order sentence φ, either $T \models \varphi$ and $T^R \models [\![\varphi]\!] = \top$, or $T \models \neg\varphi$ and $T^R \models [\![\varphi]\!] = \bot$. So $\operatorname{fdcl}_{\mathbb{B}}(\emptyset) = \{\top, \bot\}$. □

3.2. **First Order and Pointwise Definability.** To prepare the way for a characterization of the definable elements of sort $\mathbb{K}$, we introduce two auxiliary notions, one that is stronger than definability in sort $\mathbb{K}$ and one that is weaker than definability in sort $\mathbb{K}$. We will work in the nice randomization $\mathcal{P} = (\mathcal{K}, \mathcal{B})$ of $\mathcal{M}$, and let A be a subset of $\widehat{\mathcal{K}}$ and $\boldsymbol{b}$ be an element of $\widehat{\mathcal{K}}$.

Definition 3.2.1. A first order formula $\varphi(u, \vec{v})$ is *functional* if

$$T \models (\forall \vec{v})(\exists^{\leq 1} u)\varphi(u, \vec{v}).$$

We say that $\boldsymbol{b}$ is *first order definable on* E *over* A if there is a functional formula $\varphi(u, \vec{v})$ and a tuple $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$ such that $\mathsf{E} = [\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$.

We say that $\boldsymbol{b}$ is *first order definable over* A, in symbols $\boldsymbol{b} \in \operatorname{fdcl}(A)$, if $\boldsymbol{b}$ is first order definable on $\top$ over A.

Remarks 3.2.2. $\boldsymbol{b}$ is first order definable over A if and only if there is a first order formula $\varphi(u, \vec{v})$ and a tuple $\vec{\boldsymbol{a}}$ from A such that

$$\mu([\![(\forall u)(\varphi(u, \vec{\boldsymbol{a}}) \leftrightarrow u = \boldsymbol{b})]\!]) = 1.$$

First order definability has finite character, that is, $\boldsymbol{b}$ is first order definable over A if and only if $\boldsymbol{b}$ is first order definable over some finite subset of A.

If $\boldsymbol{b}$ is first order definable on E over A, then E is first order definable over $A \cup \{\boldsymbol{b}\}$.

If $\boldsymbol{b}$ is first order definable on D over A, and E is first order definable over $A \cup \{\boldsymbol{b}\}$, then $\boldsymbol{b}$ is first order definable on $\mathsf{D} \sqcap \mathsf{E}$ over A.

Lemma 3.2.3. *If $\boldsymbol{b}$ is first order definable over A then $\boldsymbol{b}$ is definable over A in $\mathcal{N}$. Thus* $\operatorname{fdcl}(A) \subseteq \operatorname{dcl}(A)$.

Proof. Let $\mathcal{N}' \succ \mathcal{N}$ and suppose that $\operatorname{tp}^{\mathcal{N}'}(\boldsymbol{b}) = \operatorname{tp}^{\mathcal{N}'}(\boldsymbol{d})$. Then

$$[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] = [\![\varphi(\boldsymbol{d}, \vec{\boldsymbol{a}})]\!] = \top.$$

Since φ is functional,

$$[\![(\forall t)(\forall u)(\varphi(t, \vec{\boldsymbol{a}}) \wedge \varphi(u, \vec{\boldsymbol{a}}) \rightarrow t = u)]\!] = \top.$$

Then $[\![\boldsymbol{b} = \boldsymbol{d}]\!] = \top$, so $\boldsymbol{b} = \boldsymbol{d}$, and by Fact 2.2.1, $\boldsymbol{b} \in \operatorname{dcl}(A)$. $\square$

Definition 3.2.4. When A is countable, we define

$$[\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!] := \{\omega \in \Omega \colon b(\omega) \in \operatorname{dcl}^{\mathcal{M}}(A(\omega))\}.$$

Lemma 3.2.5. *If A is countable, then*

$$[\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!] = \bigcup\{[\![\theta(b, \vec{a})]\!] \colon \theta(u, \vec{v}) \textit{ functional, } \vec{\boldsymbol{a}} \in A^{<\mathbb{N}}\},$$

and $[\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!] \in \mathcal{B}$.

Proof. Note that for every first order formula $\theta(u, \vec{v})$, the formula

$$\theta(u, \vec{v}) \wedge (\exists^{\leq 1} u)\, \theta(u, \vec{v})$$

is functional. Therefore $\omega \in [\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!]$ if and only if $b(\omega) \in \operatorname{dcl}^{\mathcal{M}}(A(\omega))$, and this holds if and only if there is a functional formula $\theta(u, \vec{v})$ and a tuple $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$ such that $\mathcal{M} \models \theta(b(\omega), \vec{a}(\omega))$. Since A and L are countable, $[\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!]$ is the union of countably many events in $\mathcal{B}$, and thus belongs to $\mathcal{B}$. $\square$

Definition 3.2.6. When A is countable, we say that $\boldsymbol{b}$ is *pointwise definable over A* if

$$\mu([\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!]) = 1.$$

Corollary 3.2.7. *If A is countable, then $\boldsymbol{b}$ is pointwise definable over A if and only if there is a function f on Ω such that:*

1. *For each $\omega \in \Omega$, $f(\omega)$ is a pair $\langle \theta_\omega(u, \vec{v}), \vec{a}_\omega \rangle$ where $\theta_\omega(u, \vec{v})$ is functional and $\vec{a}_\omega \in A^{|\vec{v}|}$;*
2. *f is $\sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$-measurable (i.e., the inverse image of each point belongs to $\sigma(\operatorname{fdcl}_{\mathbb{B}}(A))$);*
3. *$\mathcal{M} \models \theta_\omega(b(\omega), \vec{a}_\omega(\omega))$ for almost every $\omega \in \Omega$.*

Proof. If $\omega \in [\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!]$, let $f(\omega)$ be the first pair $\langle \theta_\omega, \vec{a}_\omega \rangle$ such that $\theta_\omega(u, \vec{v})$ is functional, $\vec{a}_\omega \in A^{|\vec{v}|}$, and $\mathcal{M} \models \theta_\omega(b(\omega), \vec{a}_\omega(\omega))$. Otherwise let $f(\omega) = \langle \bot, \emptyset \rangle$. The result then follows from Lemma 3.2.5. $\square$

Lemma 3.2.8. *If $\boldsymbol{b}$ is definable over A in $\mathcal{N}$, then $\boldsymbol{b}$ is pointwise definable over some countable subset of A.*

Proof. By Fact 2.2.1 (3), we may assume that A is countable. By Lemma 3.2.5, the measure $r := \mu([\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!])$ exists. Suppose $\boldsymbol{b}$ is not pointwise definable over A. Then $r < 1$. For each finite collection $\chi_1(u, \vec{v}), \ldots, \chi_n(u, \vec{v})$ of first order formulas, each tuple $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$, and each $\omega \in \Omega \setminus [\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!]$, the sentence

$$(\exists u)[u \neq b(\omega) \wedge \bigwedge_{i=1}^{n} [\chi_i(b(\omega), \vec{a}(\omega)) \leftrightarrow \chi_i(u, \vec{a}(\omega))]$$

holds in $\mathcal{M}$, because $b(\omega)$ is not definable over $A(\omega)$. Therefore in $\mathcal{P}$ we have

$$\mu[\![(\exists u)[u \neq b \wedge \bigwedge_{i=1}^{n} [\chi_i(b, \vec{a}) \leftrightarrow \chi_i(u, \vec{a})]]\!] \geq 1 - r.$$

By condition 2.1.1 (5), there is an element $\boldsymbol{d} \in \widehat{\mathcal{K}}$ such that

$$\mu[\![d \neq b \wedge \bigwedge_{i=1}^{n} [\chi_i(b, \vec{a}) \leftrightarrow \chi_i(d, \vec{a})]]\!] \geq 1 - r.$$

It follows that $\mu([\![d \neq b]\!]) \geq 1 - r$, and $[\![\chi_i(b, \vec{a})]\!] \doteq [\![\chi_i(d, \vec{a})]\!]$ for each $i \leq n$. By compactness, in some elementary extension of $\mathcal{N}$ there is an element $\boldsymbol{d}$ such that $\mu[\![\boldsymbol{d} \neq \boldsymbol{b}]\!] \geq 1 - r$, and $[\![\chi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] = [\![\chi(\boldsymbol{d}, \vec{\boldsymbol{a}})]\!]$ for each first order formula $\chi(u, \vec{v})$. Then $\boldsymbol{d} \neq \boldsymbol{b}$, and by quantifier elimination, $\operatorname{tp}(\boldsymbol{d}/A) = \operatorname{tp}(\boldsymbol{b}/A)$. Hence by Fact 2.2.1 (2), $\boldsymbol{b} \notin \operatorname{dcl}(A)$. $\square$

The following example shows that the converse of Lemma 3.2.8 fails badly.

Example 3.2.9. Let $\mathcal{M}$ be a finite structure with a constant symbol for every element. Then every element of $\mathcal{K}$ is pointwise definable without parameters, but the only elements of $\widehat{\mathcal{K}}$ that are definable without parameters are the equivalence classes of constant functions $b \colon \Omega \to \mathcal{M}$.

3.3. **Definability in Sort $\mathbb{K}$.** We will now give necessary and sufficient conditions for an element of $\boldsymbol{b} \in \widehat{\mathcal{K}}$ to be definable over a parameter set $A \subseteq \widehat{K}$ in $\mathcal{N}$.

Theorem 3.3.1. *$\boldsymbol{b}$ is definable over A if and only if there exist pairwise disjoint events $\{\mathsf{E}_n \colon n \in \mathbb{N}\}$ such that $\sum_{n \in \mathbb{N}} \mu(\mathsf{E}_n) = 1$, and for each n, E_n is definable over A, and $\boldsymbol{b}$ is first order definable on E_n over A.*

Proof. ($\Rightarrow$): Suppose $\boldsymbol{b} \in \operatorname{dcl}(A)$. By Lemma 3.2.8, $\boldsymbol{b}$ is pointwise definable over some countable subset A_0 of A. The set of all events C such that $\boldsymbol{b}$ is first order definable on C over A_0 is countable, and may be arranged in a list $\{\mathsf{C}_n \colon n \in \mathbb{N}\}$. Let $\mathsf{E}_0 = \mathsf{C}_0$, and

$$\mathsf{E}_{n+1} = \mathsf{C}_{n+1} \sqcap \neg(\mathsf{C}_0 \sqcup \cdots \sqcup \mathsf{C}_n).$$

The events E_n are pairwise disjoint, and for each n we have

$$\mathsf{E}_0 \sqcup \cdots \sqcup \mathsf{E}_n = \mathsf{C}_0 \sqcup \cdots \sqcup \mathsf{C}_n.$$

By Remarks 3.2.2, for each n, $\boldsymbol{b}$ is first order definable on E_n over A. By Lemma 3.2.5 and pointwise definability,

$$\sum_{n\in\mathbb{N}} \mu(\mathsf{E}_n) = \lim_{n\to\infty} \mu(\mathsf{C}_0 \sqcup \cdots \sqcup \mathsf{C}_n) = \mu([\![\mathrm{dcl}^{\mathcal{M}}(A_0)]\!]) = 1.$$

By Remarks 3.2.2, E_n is definable over $A \cup \{\boldsymbol{b}\}$, and since $\boldsymbol{b}$ is definable over A, E_n is definable over A by Fact 2.2.3.

($\Leftarrow$): Let E_n be as in the theorem. For each n, we have $\mathsf{E}_n = [\![\theta_n(\boldsymbol{b}, \vec{\boldsymbol{a}}_n)]\!]$ for some functional formula θ_n and tuple $\vec{\boldsymbol{a}}_n \in A^{<\mathbb{N}}$. Since E_n is definable over A, by Theorem 3.1.2 there is a sequence of formulas $\psi_k(\vec{v})$ and tuples $\vec{\boldsymbol{a}_k} \in A^{<\mathbb{N}}$ such that

$$\lim_{k\to\infty} d_{\mathbb{B}}([\![\psi_k(\vec{\boldsymbol{a}}_k)]\!], [\![\theta_n(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]) = 0.$$

Suppose $\boldsymbol{d}$ has the same type over A as $\boldsymbol{b}$ in some elementary extension $\mathcal{N}'$ of $\mathcal{N}$. Then

$$\lim_{k\to\infty} d_{\mathbb{B}}([\![\psi_k(\vec{\boldsymbol{a}}_k)]\!], [\![\theta_n(\boldsymbol{d}, \vec{\boldsymbol{a}})]\!]) = 0.$$

Hence

$$[\![\theta_n(\boldsymbol{d}, \vec{\boldsymbol{a}}_n)]\!] = [\![\theta_n(\boldsymbol{b}, \vec{\boldsymbol{a}}_n)]\!] = \mathsf{E}_n$$

in $\mathcal{N}'$. Since $\theta_n(u, \vec{v})$ is functional, we have $[\![\theta_n(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \sqsubseteq [\![\boldsymbol{d} = \boldsymbol{b}]\!]$ for each n. Then

$$\mu([\![\boldsymbol{d} = \boldsymbol{b}]\!]) \geq \sum_{n\in\mathbb{N}} \mu(\mathsf{E}_n) = 1,$$

so $\boldsymbol{d} = \boldsymbol{b}$. Then by Fact 2.2.1, $\boldsymbol{b} \in \mathrm{dcl}(A)$. □

Corollary 3.3.2. *An element $\boldsymbol{b} \in \widehat{\mathcal{K}}$ is definable without parameters if and only if $\boldsymbol{b}$ is first order definable without parameters. Thus* $\mathrm{dcl}(\emptyset) = \mathrm{fdcl}(\emptyset)$.

Proof. ($\Rightarrow$): Suppose $\boldsymbol{b} \in \mathrm{dcl}(\emptyset)$. By Theorem 3.3.1, there is an event E such that $\mu(\mathsf{E}) > 0$, E is definable without parameters, and $\boldsymbol{b}$ is first order definable on E without parameters. By Corollary 3.1.3 we have $\mathsf{E} = \top$, so $\boldsymbol{b}$ is first order definable without parameters.

($\Leftarrow$): By Lemma 3.2.3. □

Corollary 3.3.3. *If* $\mathrm{fdcl}_{\mathbb{B}}(A)$ *is finite, then* $\mathrm{dcl}_{\mathbb{B}}(A) = \mathrm{fdcl}_{\mathbb{B}}(A)$ *and* $\mathrm{dcl}(A) = \mathrm{fdcl}(A)$.

Proof. $\mathrm{dcl}_{\mathbb{B}}(A) = \mathrm{fdcl}_{\mathbb{B}}(A)$ follows from Theorem 3.1.2. Lemma 3.2.3 gives $\mathrm{dcl}(A) \supseteq \mathrm{fdcl}(A)$. For the other inclusion, suppose $\boldsymbol{b} \in \mathrm{dcl}(A)$. By Theorem 3.3.1, there is a finite partition $\mathsf{E}_0, \ldots, \mathsf{E}_k$ of $\top$, a tuple $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$, and first order formulas $\psi_i(\vec{v})$ such that $\mathsf{E}_i = [\![\psi_i(\vec{\boldsymbol{a}})]\!]$ and $\boldsymbol{b}$ is first order definable on E_i. Then there are functional formulas $\varphi_i(u, \vec{v})$ such that $\mathsf{E}_i \doteq [\![\varphi_i(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$. We may take the formulas $\psi_i(\vec{v})$ to be pairwise inconsistent and such that

$T \models \bigvee_{i=0}^{n} \psi(\vec{v})$. Then $\bigwedge_{i=0}^{n}(\psi_i(\vec{v}) \to \varphi_i(u, \vec{v}))$ is a functional formula such that

$$[\![\bigwedge_{i=0}^{n}(\psi_i(\vec{\boldsymbol{a}}) \to \varphi_i(\boldsymbol{b}, \vec{\boldsymbol{a}}))]\!] = \top,$$

so $\boldsymbol{b}$ is first order definable over A. □

Corollary 3.3.4. *$\boldsymbol{b}$ is definable over A if and only if:*

(1) *$\boldsymbol{b}$ is pointwise definable over some countable subset of A;*
(2) *for each functional formula $\varphi(u, \vec{v})$ and tuple $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$, $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$ is definable over A.*

Proof. (⇒): Suppose $\boldsymbol{b} \in \operatorname{dcl}(A)$. Then (1) holds by Lemma 3.2.8. $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$ is obviously definable over $A \cup \{\boldsymbol{b}\}$, so $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$ is definable over A by Fact 2.2.3, and thus (2) holds.

(⇐): Assume conditions (1) and (2). By (1) and Lemma 3.2.5, there is a sequence of functional formulas $\theta_n(u, \vec{v})$ and tuples $\vec{\boldsymbol{a}}_n \in A^{<\mathbb{N}}$ such that

$$[\![b \in \operatorname{dcl}^{\mathcal{M}}(A)]\!] = \bigcup_{n\in\mathbb{N}} [\![\theta_n(b, \vec{a}_n)]\!] \doteq \Omega.$$

Let $\mathsf{E}_n = [\![\theta_n(\boldsymbol{b}, \vec{\boldsymbol{a}}_n)]\!]$, so $\boldsymbol{b}$ is first order definable on E_n over A. By Remark 3.2.2, we may take the E_n to be pairwise disjoint, and thus $\sum_{n\in\mathbb{N}} \mu(\mathsf{E}_n) = 1$. By (2), E_n is definable over A for each n. Then by Theorem 3.3.1, $\boldsymbol{b} \in \operatorname{dcl}(A)$. □

Corollary 3.3.5. *$\boldsymbol{b}$ is definable over A if and only if:*

(1) *b is pointwise definable over some countable subset of A;*
(2) $\operatorname{fdcl}_{\mathbb{B}}(A \cup \{\boldsymbol{b}\}) \subseteq \operatorname{dcl}_{\mathbb{B}}(A)$.

Theorem 3.3.6. *$\boldsymbol{b}$ is definable over A if and only if $\boldsymbol{b} = \lim_{m\to\infty} \boldsymbol{b}_m$, where each $\boldsymbol{b}_m$ is first-order definable over A. Thus* $\operatorname{dcl}(A) = \operatorname{cl}(\operatorname{fdcl}(A))$.

Proof. (⇒): Suppose that $\boldsymbol{b} \in \operatorname{dcl}(A)$. If A is empty, then $\boldsymbol{b}$ is already first order definable from A by Corollary 3.3.2. Assume A is not empty and let $\boldsymbol{c} \in A$. Let $\{\mathsf{E}_n : n \in \mathbb{N}\}$ be as in Theorem 3.3.1, and fix an $\varepsilon > 0$. Then for some n, $\sum_{k=0}^{n} \mu(\mathsf{E}_k) > 1 - \varepsilon$. For each k, E_k is definable over A, so by Theorem 3.1.2, there is an event $\mathsf{D}_k \in \operatorname{fdcl}_{\mathbb{B}}(A)$ such that $\mu(\mathsf{D}_k \triangle \mathsf{E}_k) < \varepsilon/n$. Since the events E_k are pairwise disjoint, we may also take the events D_k to be pairwise disjoint. We have $\mathsf{E}_k = [\![\theta_k(\boldsymbol{b}, \vec{\boldsymbol{a}}_k)]\!]$ for some functional $\theta_k(u, \vec{v})$, so we may assume that D_k has the additional properties that $\mathsf{D}_k \sqsubseteq [\![(\exists! u)\theta_k(u, \vec{\boldsymbol{a}}_k)]\!]$, and that $\mathsf{D}_k = [\![\psi_k(\vec{\boldsymbol{a}}_k)]\!]$ for some formula $\psi_k(\vec{v})$. Then there is a unique element $\boldsymbol{d} \in \widehat{\mathcal{K}}$ such that

$$\begin{cases} \mathcal{M} \models \theta_k(d(\omega), \vec{a}_k(\omega)) & \text{if } k \le n \text{ and } \omega \in [\![\psi_k(\vec{a}_k)]\!], \\ d(\omega) = c(\omega) & \text{if } \omega \in \Omega \setminus \bigcup_{k=0}^{n} [\![\psi_k(\vec{a}_k)]\!]. \end{cases}$$

Then $\boldsymbol{d}$ is first order definable over A, and $d_{\mathbb{K}}(\boldsymbol{b}, \boldsymbol{d}) < \varepsilon$.

(⇐): This follows because first order definability implies definability (Lemma 3.2.3) and the set $\operatorname{dcl}(A)$ is metrically closed (Fact 2.2.3 (2)). □

The following result was proved in [Be] by an indirect argument using Lascar types. We give a simple direct proof here.

Proposition 3.3.7. *For any model $\mathcal{N} = (\widehat{\mathcal{K}}, \widehat{\mathcal{B}})$ of T^R and set $A \subseteq \widehat{\mathcal{K}}$, $\operatorname{acl}_{\mathbb{B}}(A) = \operatorname{dcl}_{\mathbb{B}}(A)$ and $\operatorname{acl}(A) = \operatorname{dcl}(A)$.*

Proof. By Facts 2.2.1 and 2.2.2, we may assume $\mathcal{N}$ is $\aleph_1$-saturated and A is countable. Suppose an event $\mathsf{E} \in \widehat{\mathcal{B}}$ is not definable over A. By Fact 2.2.1 and $\aleph_1$-saturation there exists $\mathsf{D} \in \widehat{\mathcal{B}}$ such that $\operatorname{tp}(\mathsf{D}/A) = \operatorname{tp}(\mathsf{E}/A)$ but $d_{\mathbb{B}}(\mathsf{D}, \mathsf{E}) > 0$. By $\aleph_1$-saturation again, there is a countable sequence of events $\langle \mathsf{F}_n \colon n \in \mathbb{N} \rangle$ in $\widehat{\mathcal{B}}$ such that

$$\mu(\mathsf{C} \cap \mathsf{F}_n) = \mu(\mathsf{C} \setminus \mathsf{F}_n) = \mu(\mathsf{C})/2$$

for each n and each event C in the Boolean algebra generated by

$$\operatorname{fdcl}_{\mathbb{B}}(A) \cup \{\mathsf{D}, \mathsf{E}\} \cup \{\mathsf{F}_k \colon k < n\}.$$

For each n, let

$$\mathsf{D}_n = (\mathsf{D} \cap \mathsf{F}_n) \cup (\mathsf{E} \setminus \mathsf{F}_n).$$

Then for each $\mathsf{C} \in \operatorname{fdcl}_{\mathbb{B}}(A)$ and $n \in \mathbb{N}$, we have

$$\mu(\mathsf{D}_n \cap \mathsf{C}) = \mu(\mathsf{D} \cap \mathsf{C})/2 + \mu(\mathsf{E} \cap \mathsf{C})/2 = \mu(\mathsf{E} \cap \mathsf{C}).$$

By quantifier elimination, $\operatorname{tp}(\mathsf{D}_n/A) = \operatorname{tp}(\mathsf{E}/A)$ for each $n \in \mathbb{N}$. Moreover, whenever $k < n$ we have

$$\mathsf{D}_n \setminus \mathsf{D}_k = ((\mathsf{D} \setminus \mathsf{D}_k) \cap \mathsf{F}_n) \cup ((\mathsf{E} \setminus \mathsf{D}_k) \setminus \mathsf{F}_n),$$

so

$$\mu(\mathsf{D}_n \setminus \mathsf{D}_k) = \mu(\mathsf{D} \setminus \mathsf{D}_k)/2 + \mu(\mathsf{E} \setminus \mathsf{D}_k)/2.$$

Note that whenever $\operatorname{tp}(\mathsf{D}'/A) = \operatorname{tp}(\mathsf{D}''/A)$, we have $\mu(\mathsf{D}') = \mu(\mathsf{D}'')$, and hence

$$\mu(\mathsf{D}' \setminus \mathsf{D}'') = \mu(\mathsf{D}'' \setminus \mathsf{D}') = d_{\mathbb{B}}(\mathsf{D}', \mathsf{D}'')/2.$$

Therefore

$$d_{\mathbb{B}}(\mathsf{D}_n, \mathsf{D}_k) = d_{\mathbb{B}}(\mathsf{D}, \mathsf{D}_k)/2 + d_{\mathbb{B}}(\mathsf{E}, \mathsf{D}_k)/2 \geq d_{\mathbb{B}}(\mathsf{D}, \mathsf{E})/2.$$

It follows that the set of realizations of $\operatorname{tp}(\mathsf{E}/A)$ is not compact, and E is not algebraic over A. This shows that $\operatorname{acl}_{\mathbb{B}}(A) = \operatorname{dcl}_{\mathbb{B}}(A)$.

Now suppose $\boldsymbol{b} \in \operatorname{acl}(A) \setminus \operatorname{dcl}(A)$. There is an element $\boldsymbol{c} \in \widehat{\mathcal{K}}$ such that $\operatorname{tp}(\boldsymbol{b}/A) = \operatorname{tp}(\boldsymbol{c}/A)$ but $d_{\mathbb{K}}(\boldsymbol{b}, \boldsymbol{c}) > 0$. For each first order formula $\psi(u, \vec{v})$ and $\vec{\boldsymbol{a}} \in A^{<\mathbb{N}}$, $[\![\psi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \operatorname{acl}_{\mathbb{B}}(\{\boldsymbol{b}\} \cup A) \subseteq \operatorname{acl}_{\mathbb{B}}(\operatorname{acl}(A))$. By Fact 2.2.3, $[\![\psi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \operatorname{acl}_{\mathbb{B}}(A)$. By the preceding paragraph, $[\![\psi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \operatorname{dcl}_{\mathbb{B}}(A)$. Since $\operatorname{tp}(\boldsymbol{b}/A) = \operatorname{tp}(\boldsymbol{c}/A)$, we have $\operatorname{tp}([\![\psi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]/A) = \operatorname{tp}([\![\psi(\boldsymbol{c}, \vec{\boldsymbol{a}})]\!]/A)$. By Fact 2.2.1, it follows that $[\![\psi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] = [\![\psi(\boldsymbol{c}, \vec{\boldsymbol{a}})]\!]$ for every first order formula $\psi(u, \vec{v})$. Then $\operatorname{tp}(b(\omega)/A(\omega)) = \operatorname{tp}(c(\omega)/A(\omega))$ for μ-almost all ω. By $\aleph_1$-saturation, there are countably many independent events $\mathsf{D}_n \in \widehat{\mathcal{B}}$ such that $\mathsf{D}_n \sqsubseteq [\![\boldsymbol{b} \neq \boldsymbol{c}]\!]$ and $\mu(\mathsf{D}_n) = d_{\mathbb{K}}(\boldsymbol{b}, \boldsymbol{c})/2$. Let $\boldsymbol{c}_n$ agree with $\boldsymbol{c}$ on D_n and agree with $\boldsymbol{b}$ elsewhere. We have $\operatorname{tp}(\boldsymbol{c}_n/A) = \operatorname{tp}(\boldsymbol{b}/A)$ for every $n \in \mathbb{N}$, and

$d_{\mathbb{K}}(\boldsymbol{c}_n, \boldsymbol{c}_k) = d_{\mathbb{K}}(\boldsymbol{b}, \boldsymbol{c})/2$ whenever $k < n$. Thus the set of realizations of $\mathrm{tp}(\boldsymbol{b}/A)$ is not compact, contradicting the fact that $\boldsymbol{b} \in \mathrm{acl}(A)$. □

4. A Special Case: $\aleph_0$-categorical theories

4.1. **Definability and $\aleph_0$-Categoricity.**

We use our preceding results to characterize $\aleph_0$-categorical theories in terms of definability in randomizations.

Theorem 4.1.1. *The following are equivalent:*

1. T *is* $\aleph_0$*-categorical;*
2. $\mathrm{fdcl}_{\mathbb{B}}(A)$ *is finite for every finite* A*;*
3. $\mathrm{dcl}_{\mathbb{B}}(A)$ *is finite for every finite* A*;*
4. $\mathrm{fdcl}_{\mathbb{B}}(A) = \mathrm{dcl}_{\mathbb{B}}(A)$ *for every finite* A*;*
5. $\mathrm{fdcl}(A)$ *is finite for every finite* A*;*
6. $\mathrm{dcl}(A)$ *is finite for every finite* A*.*
7. $\mathrm{fdcl}(A) = \mathrm{dcl}(A)$ *for every finite* A*;*

Proof. By the Ryll-Nardzewski Theorem (see [CK], Theorem 2.3.13), (1) is equivalent to

(0) For each n there are only finitely many formulas in n variables up to T-equivalence.

Assume (0) and let $A \subseteq \widehat{\mathcal{K}}$ be finite. Then (2) holds. Moreover, there are only finitely many functional formulas in $|A| + 1$ variables, so (5) holds. Then by Corollary 3.3.3, (3), (4), (6), and (7) hold.

Now assume that (0) fails.

Proof that (2) and (3) fail: For some n there are infinitely many formulas in n variables that are not T-equivalent. Hence there is an n-type p in T without parameters that is not isolated. So there are formulas $\varphi_1(\vec{v}), \varphi_2(\vec{v}), \ldots$ in p such that for each $k > 0$, $T \models \varphi_{k+1} \to \varphi_k$ but the formula $\theta_k = \varphi_k \wedge \neg\varphi_{k+1}$ is consistent with T. The formulas θ_k are consistent but pairwise inconsistent. By Fullness, for each $k > 0$ there exists an n-tuple $\vec{\boldsymbol{b}}_k \in \widehat{\mathcal{K}}^n$ such that $[\![\theta_k(\vec{\boldsymbol{b}}_k)]\!] = \top$. Since the measured algebra $(\widehat{\mathcal{B}}, \mu)$ is atomless, there are pairwise disjoint events $\mathsf{E}_1, \mathsf{E}_2, \ldots$ in $\widehat{\mathcal{B}}$ such that $\mu(\mathsf{E}_k) = 2^{-k}$ for each $k > 0$. By applying Lemma 2.1.8 k times, we see that for each $k > 0$ there is an n-tuple $\vec{\boldsymbol{a}}_k \in \widehat{\mathcal{K}}^n$ that agrees with $\vec{\boldsymbol{b}}_i$ on E_i whenever $0 < i \le k$. Whenever $0 < k \le j$, we have $\mu([\![\vec{\boldsymbol{a}}_k = \vec{\boldsymbol{a}}_j]\!]) \ge 1 - 2^{-k}$. So $\langle \vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_2, \ldots\rangle$ is a Cauchy sequence, and by metric completeness the limit $\vec{\boldsymbol{a}} = \lim_{k\to\infty} \vec{\boldsymbol{a}}_k$ exists in $\widehat{\mathcal{K}}^n$. Let $A = \mathrm{range}(\vec{\boldsymbol{a}})$. For each $k > 0$ we have $\mathsf{E}_k = [\![\vec{\boldsymbol{a}} = \vec{\boldsymbol{b}}_k]\!] = [\![\theta_k(\vec{\boldsymbol{a}})]\!]$, so $\mathsf{E}_k \in \mathrm{fdcl}_{\mathbb{B}}(A)$. Thus $\mathrm{fdcl}_{\mathbb{B}}(A)$ is infinite, so (2) fails and (3) fails.

Proof that (4) fails: Let E_k be as in the preceding paragraph. The set $\mathrm{fdcl}_{\mathbb{B}}(A)$ is countable. But the closure $\mathrm{cl}(\mathrm{fdcl}_{\mathbb{B}}(A))$ is uncountable, because for each set $S \subseteq \mathbb{N} \setminus \{0\}$, the supremum $\bigsqcup_{k\in S} \mathsf{E}_k$ belongs to $\mathrm{cl}(\mathrm{fdcl}_{\mathbb{B}}(A))$. Thus by Theorem 3.1.2,

$$\mathrm{dcl}_{\mathbb{B}}(A) = \mathrm{cl}(\mathrm{fdcl}_{\mathbb{B}}(A)) \neq \mathrm{fdcl}_{\mathbb{B}}(A),$$

and (4) fails.

Proof that (5), (6), and (7) fail: By Corollary 2.1.6, there exist $\boldsymbol{c}, \boldsymbol{d} \in \mathcal{K}$ such that $[\![\boldsymbol{c} \neq \boldsymbol{d}]\!] = \top$. Let C be the finite set $C = A \cup \{\boldsymbol{c}, \boldsymbol{d}\}$. By Remark 2.2.4, for any event $\mathsf{D} \in \mathrm{fdcl}_{\mathbb{B}}(A)$, the characteristic function 1_{D} of D with respect to $\boldsymbol{c}, \boldsymbol{d}$ is definable over C. Moreover, we always have $d_{\mathbb{K}}(1_{\mathsf{D}}, 1_{\mathsf{E}}) = d_{\mathbb{B}}(\mathsf{D}, \mathsf{E})$. It follows that $\mathrm{fdcl}(C)$ is infinite, so (5) and (6) fail. To show that (7) fails, we take an event $\mathsf{D} \in \mathrm{dcl}_{\mathbb{B}}(A) \setminus \mathrm{fdcl}_{\mathbb{B}}(A)$. By Theorem 3.1.2 we have $\mathsf{D} \in \mathrm{cl}(\mathrm{fdcl}_{\mathbb{B}}(A))$. It follows that $1_{\mathsf{D}} \in \mathrm{cl}(\mathrm{fdcl}(C))$, so by Theorem 3.3.6, $1_{\mathsf{D}} \in \mathrm{dcl}(C)$. Hence $\mathrm{dcl}(C)$ is uncountable. But $\mathrm{fdcl}(C)$ is countable, so (7) fails. $\square$

By the Ryll-Nardzewski Theorem, if T is $\aleph_0$-categorical then for each n, T has finitely many n-types; so each type p in the variables $(u, \vec{v})$ has an *isolating formula*, that is, a formula $\varphi(u, \vec{v})$ such that $T \models \varphi(u, \vec{v}) \leftrightarrow \bigwedge p$.

We now characterize the definable closure of a finite set $A \subseteq \widehat{\mathcal{K}}$ in the case that T is $\aleph_0$-categorical. Hereafter, when A is a finite subset of $\widehat{\mathcal{K}}$, $\vec{\boldsymbol{a}}$ will denote a finite tuple whose range is A.

Corollary 4.1.2. *Suppose that T is $\aleph_0$-categorical, $\boldsymbol{b} \in \widehat{\mathcal{K}}$, and A is a finite subset of $\widehat{\mathcal{K}}$. Then $\boldsymbol{b} \in \mathrm{dcl}(A)$ if and only if:*

1. *$\boldsymbol{b}$ is pointwise definable over A;*
2. *for every isolating formula $\varphi(u, \vec{v})$, if $\mu([\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]) > 0$ then*
$$[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] = [\![(\exists u)\varphi(u, \vec{\boldsymbol{a}})]\!].$$

Proof. ($\Rightarrow$): Suppose $\boldsymbol{b} \in \mathrm{dcl}(A)$. (1) holds by Lemma 3.2.8. Suppose $\varphi(u, \vec{v})$ is isolating and $\mu([\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]) > 0$. We have $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \mathrm{fdcl}_{\mathbb{B}}(\{\boldsymbol{b}\} \cup A)$, so by Corollary 3.3.5, $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \mathrm{dcl}_{\mathbb{B}}(A)$. By Theorem 4.1.1, $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \mathrm{fdcl}_{\mathbb{B}}(A)$. We note that $(\exists u)\varphi(u, \vec{v})$ is an isolating formula, so $[\![(\exists u)\varphi(u, \vec{\boldsymbol{a}})]\!]$ is an atom of $\mathrm{fdcl}_{\mathbb{B}}(A)$. Therefore (2) holds.

($\Leftarrow$): Assume (1) and (2). By (2), for every isolating formula $\varphi(u, \vec{v})$ such that $\mu([\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]) > 0$, we have
$$[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] \in \mathrm{fdcl}_{\mathbb{B}}(A).$$
Every formula $\theta(u, \vec{v})$ is T-equivalent to a finite disjunction of isolating formulas in the variables $(u, \vec{v})$. It follows that $\mathrm{fdcl}_{\mathbb{B}}(A \cup \{\boldsymbol{b}\}) \subseteq \mathrm{fdcl}_{\mathbb{B}}(A)$. Therefore by Corollary 3.3.5, $\boldsymbol{b} \in \mathrm{dcl}(A)$. $\square$

Corollary 4.1.3. *Suppose that T is $\aleph_0$-categorical, $\boldsymbol{b} \in \widehat{\mathcal{K}}$, and A is a finite subset of $\widehat{\mathcal{K}}$. Then $\boldsymbol{b} \in \mathrm{dcl}(A)$ if and only if for every isolating formula $\psi(\vec{v})$ there is a functional formula $\varphi(u, \vec{v})$ such that $[\![\psi(\vec{\boldsymbol{a}})]\!] \sqsubseteq [\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$.*

Proof. ($\Rightarrow$): Suppose $\boldsymbol{b} \in \mathrm{dcl}(A)$. By Theorem 4.1.1, $\boldsymbol{b}$ is first order definable over $\vec{\boldsymbol{a}}$, so there is a functional formula $\varphi(u, \vec{v})$ such that $[\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!] = \top$. Then for every isolating $\psi(\vec{v})$ we have $[\![\psi(\vec{\boldsymbol{a}})]\!] \sqsubseteq [\![\varphi(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$.

($\Leftarrow$): There is a finite set $\{\psi_0(\vec{v}), \ldots, \psi_k(\vec{v})\}$ that contains exactly one isolating formula for each $|\vec{\boldsymbol{a}}|$-type of T. By hypothesis, for each $i \leq k$ there is a functional formula $\varphi_i(u, \vec{v})$ such that $[\![\psi_i(\vec{\boldsymbol{a}})]\!] \sqsubseteq [\![\varphi_i(\boldsymbol{b}, \vec{\boldsymbol{a}})]\!]$. Since the formulas $\psi_i(\vec{v})$ are pairwise inconsistent, the formula $\bigvee_{i=0}^k(\psi_i(\vec{v}) \wedge \varphi_i(u,\vec{v}))$ is functional, and

$$[\![\bigvee_{i=0}^{k}(\psi_i(\vec{\boldsymbol{a}}) \wedge \varphi_i(\boldsymbol{b},\vec{\boldsymbol{a}}))]\!] = \top.$$

Hence $\boldsymbol{b}$ is first order definable over $\vec{\boldsymbol{a}}$, so by Lemma 3.2.3 we have $\boldsymbol{b} \in \operatorname{dcl}(A)$. □

4.2. **The Theory** DLO^R**.** We will use Corollary 4.1.3 to give a more natural characterization of the definable closure of a finite parameter set in a model of DLO^R, where DLO is the theory of dense linear order without endpoints. Note that in DLO, every type in $(v_1,\dots,v_n)$ has an isolating formula of the form $\bigwedge_{i=1}^{n-1} u_i\alpha_i u_{i+1}$ where $\{u_1,\dots u_n\} = \{v_1,\dots,v_n\}$ and each $\alpha_i \in \{<,=\}$. (This formula linearly orders the equality-equivalence classes).

Corollary 4.2.1. *Let $T = \mathrm{DLO}$, $\boldsymbol{b} \in \widehat{\mathcal{K}}$, and A be a finite subset of $\widehat{\mathcal{K}}$. Then $\boldsymbol{b} \in \operatorname{dcl}(A)$ if and only if for every isolating formula $\psi(v_1,\dots,v_n)$ there is an $i \in \{1,\dots,n\}$ such that $[\![\psi(\vec{\boldsymbol{a}})]\!] \sqsubseteq [\![\boldsymbol{b} = \boldsymbol{a}_i]\!]$.*

Proof. For any $\mathcal{M} \models \mathrm{DLO}$ and parameter set A, we have $\operatorname{dcl}^{\mathcal{M}}(A) = A$. Therefore for every isolating formula $\psi(v_1,\dots,v_n)$ and functional formula $\varphi(u,v_1,\dots,v_n)$ there exists $i \in \{1,\dots,n\}$ such that

$$\mathrm{DLO} \models (\psi(v_1,\dots,v_n) \wedge \varphi(u,v_1,\dots,v_n)) \rightarrow u = v_i.$$

The result now follows from Corollary 4.1.3. □

In the theory DLO, we define $\min(u,v)$ and $\max(u,v)$ in the usual way. For $\boldsymbol{a},\boldsymbol{b} \in \widehat{\mathcal{K}}$, we let $\min(\boldsymbol{a},\boldsymbol{b})$ be the unique element $\boldsymbol{e} \in \widehat{\mathcal{K}}$ such that

$$[\![e = \min(a,b)]\!] = \top,$$

and similarly for max. For finite subsets A of $\widehat{\mathcal{K}}$, $\min(A)$ and $\max(A)$ are defined by repeating the two-variable functions min and max in the natural way.

We next show that in DLO^R, the definable closure of a finite set can be characterized as the closure under a "choosing function" of four variables.

Definition 4.2.2. In the theory DLO, let ℓ be the function of four variables defined by the condition

$$\ell(u,v,x,y) = x \text{ if } u < v, \text{ and } \ell(u,v,x,y) = y \text{ if not } u < v.$$

For $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d} \in \mathcal{K}$, let $\ell(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$ be the unique element $\boldsymbol{e} \in \widehat{\mathcal{K}}$ such that $[\![e = \ell(a,b,c,d)]\!] = \top$. Given a set $A \subseteq \widehat{\mathcal{K}}$, let $\operatorname{lcl}(A)$ be the closure of A under the function ℓ.

Note that in DLO, the function ℓ is definable without parameters. In both DLO and DLO^R, $\min(u,v) = \ell(u,v,u,v)$, and $\max(u,v) = \ell(u,v,v,u)$.

Proposition 4.2.3. *Let $T = \mathrm{DLO}$. Then for every finite subset A of $\widehat{\mathcal{K}}$, $\operatorname{dcl}(A) = \operatorname{lcl}(A)$.*

Proof. It is clear that $\mathrm{lcl}(A) \subseteq \mathrm{dcl}(A)$.

We prove the other inclusion. If A is empty, the result is trivial, so we assume A is non-empty. Let $\mathbf{0} = \min(A), \mathbf{1} = \max(A)$. We have $\mathbf{0}, \mathbf{1} \in \mathrm{lcl}(A)$. Let $\Omega_0 = [\![0 < 1]\!]$. Note that $\Omega \setminus \Omega_0 = [\![0 = 1]\!]$. If $\mu(\Omega_0) = 0$, then A is a singleton, and we trivially have $\mathrm{lcl}(A) = \mathrm{dcl}(A) = A$. We may therefore assume that $\mu(\Omega_0) > 0$. To simplify notation we will instead assume that $\Omega_0 = \Omega$; the argument in the general case is similar.

In the following, all characteristic functions are understood to be with respect to $\mathbf{0}, \mathbf{1}$. Note that $\ell(\boldsymbol{a}, \boldsymbol{b}, \mathbf{0}, \mathbf{1})$ is the characteristic function of the event $[\![\boldsymbol{a} < \boldsymbol{b}]\!]$. If $\boldsymbol{d}$ is the characteristic function of an event D and $\boldsymbol{e}$ is the characteristic function of an event E, then $\ell(\boldsymbol{d}, \mathbf{1}, \mathbf{1}, \mathbf{0})$ is the characteristic function of $\neg\mathsf{D}$, $\min(\boldsymbol{d}, \boldsymbol{e})$ is the characteristic function of $\mathsf{D}\sqcap\mathsf{E}$, and $\max(\boldsymbol{d}, \boldsymbol{e})$ is the characteristic function of $\mathsf{D}\sqcup\mathsf{E}$. It follows that for every quantifier-free first order formula $\varphi(\vec{v})$ of DLO with $|\vec{v}| = |\vec{\boldsymbol{a}}|$, the characteristic function of the event $[\![\varphi(\vec{\boldsymbol{a}})]\!]$ belongs to $\mathrm{lcl}(A)$. Since DLO admits quantifier elimination, the characteristic function of every event that is first order definable over A belongs to $\mathrm{lcl}(A)$. Hence by Theorem 4.1.1, the characteristic function of every event in $\mathrm{dcl}_{\mathbb{B}}(A)$ belongs to $\mathrm{lcl}(A)$. Moreover, for every $\boldsymbol{c} \in A$ and event $\mathsf{D} \in \mathrm{dcl}_{\mathbb{B}}(A)$ with characteristic function $\boldsymbol{d}$, $\boldsymbol{c} \upharpoonright \mathsf{D} := \ell(\boldsymbol{d}, \mathbf{1}, \mathbf{0}, \boldsymbol{c})$ is the element that agrees with $\boldsymbol{c}$ on D and agrees with $\mathbf{0}$ on the complement of D, so $\boldsymbol{c} \upharpoonright \mathsf{D}$ belongs to $\mathrm{lcl}(A)$. Let $\{\mathsf{D}_1, \ldots, \mathsf{D}_n\}$ be the set of atoms of $\mathrm{dcl}_{\mathbb{B}}(A)$ (which is finite because DLO is $\aleph_0$-categorical). By Corollary 4.2.1, every element of $\mathrm{dcl}(A)$ has the form

$$\max(\boldsymbol{c}_1 \upharpoonright \mathsf{D}_1, \ldots, \boldsymbol{c}_n \upharpoonright \mathsf{D}_n)$$

for some $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in A$. Therefore $\mathrm{dcl}(A) \subseteq \mathrm{lcl}(A)$. □

Example 4.2.4. In this example we show that the exchange property fails for DLO^R, even though it holds for DLO. Thus the exchange property is not preserved under randomizations. Let $T = \mathrm{DLO}$. By Fullness, there exist elements $\boldsymbol{a}, \boldsymbol{b} \in \widehat{\mathcal{K}}$ such that $\max(\boldsymbol{a}, \boldsymbol{b}) \notin \{\boldsymbol{a}, \boldsymbol{b}\}$. Let $\boldsymbol{c} = \max(\boldsymbol{a}, \boldsymbol{b}), \boldsymbol{d} = \min(\boldsymbol{a}, \boldsymbol{b})$. It is easy to check that

$$\mathrm{dcl}(\{\boldsymbol{a}, \boldsymbol{b}\}) = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}, \quad \mathrm{dcl}(\{\boldsymbol{a}, \boldsymbol{c}\}) = \{\boldsymbol{a}, \boldsymbol{c}\}, \quad \mathrm{dcl}(\{\boldsymbol{a}\}) = \{\boldsymbol{a}\}.$$

Thus $\boldsymbol{c} \in \mathrm{dcl}(\{\boldsymbol{a}, \boldsymbol{b}\}) \setminus \mathrm{dcl}(\{\boldsymbol{a}\})$ but $\boldsymbol{b} \notin \mathrm{dcl}(\{\boldsymbol{a}, \boldsymbol{c}\})$.

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