Rosiness in Continuous Logic

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(joint work with Clifton Ealy)

UCLA

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Continuous Logic

Rosiness

The Urysohn Sphere

Metric Structures

- A (bounded) metric structure is a (bounded) complete metric space (M, d), together with distinguished elements, functions (mapping M^n into M for various n) and predicates (mapping M^n into a bounded interval in $\mathbb{R}$ for various n).
- Each function and predicate is required to be uniformly continuous.
- For the sake of simplicity, we suppose that the metric is bounded by 1 and the predicates all take values in $[0, 1]$.

Examples of Metric Structures

1. If M is a structure from classical model theory, then we can consider M as a metric structure by equipping it with the discrete metric. If $P \subseteq M^n$ is a distinguished predicate, then we consider it as a mapping $P : M^n \to \{0, 1\} \subseteq [0, 1]$ by

$$P(a) = 0 \text{ if and only if } M \models P(a).$$

2. Suppose X is a Banach space with unit ball B. Then $(B, 0_X, \| \cdot \|, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure, where $f_{\alpha,\beta} : B^2 \to B$ is given by $f(x, y) = \alpha \cdot x + \beta \cdot y$ for all scalars α and β with $|\alpha| + |\beta| \leq 1$.
3. If H is a Hilbert space with unit ball B, then $(B, 0_H, \| \cdot \|, \langle \cdot, \cdot \rangle, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure.

Bounded Continuous Signatures

- As in classical logic, a signature for continuous logic consists of constant symbols, function symbols, and predicate symbols, the latter two coming also with arities.
- New to continuous logic: For every function symbol F, the signature must specify a *modulus of uniform continuity* Δ_F, which is a function $\Delta_F : (0,1] \to (0,1]$. Likewise, a modulus of uniform continuity is specified for each predicate symbol.
- The metric d is included as a (logical) predicate in analogy with $=$ in classical logic.

L-structures

An *L-structure* is a metric structure M whose distinguished constants, functions, and predicates are interpretations of the corresponding symbols in L. Moreover, the uniform continuity of the functions and predicates is witnessed by the moduli of uniform continuity specified by L.

e.g. If P is a unary predicate symbol, then for all $\epsilon > 0$ and all $x, y \in M$, we have:

$$d(x, y) < \Delta_P(\epsilon) \Rightarrow |P(x) - P(y)| \leq \epsilon.$$

Formulae

- Terms are defined as in classical logic.
- Atomic formulae are of the form $d(t_1, t_2)$ and $P(t_1, \ldots, t_n)$ where P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms.
- Connectives: If $\varphi_1, \ldots, \varphi_n$ are formulae and $u : [0,1]^n \to [0,1]$ is any continuous function, then $u(\varphi_1, \ldots, \varphi_n)$ is a formula.
- Quantifiers: If φ is a formula, then so is $\sup_x \varphi$ and $\inf_x \varphi$. (sup "=" $\forall$ and inf "=" $\exists$)
- If $\varphi(x_1, \ldots, x_n)$ is an L-formula, M an L-structure, and $a_1, \ldots, a_n$ elements of M, then M gives a value $\varphi^M(a_1, \ldots, a_n)$, which is a number in $[0,1]$ measuring "how true" φ is when $a_1, \ldots, a_n$ are plugged in for the free variables.

Theories

- A *condition* is an expression of the form "$\varphi = 0$", where φ is a formula. If φ is a sentence, then the condition "$\varphi = 0$" is called a *closed condition*.
- An *L-theory* is a set of closed *L*-conditions.
- If *M* is an *L*-structure, then the *theory of M* is the theory

$$\text{Th}(M) := \{\text{“}\varphi = 0\text{”} \mid \varphi \text{ a sentence}, \varphi^M = 0\}.$$

- An *L*-theory is *complete* if it is of the form Th(*M*) for some *L*-structure *M*.

Examples of Complete Continuous Theories

1. Infinite Dimensional Hilbert Spaces (over $\mathbb{R}$)
2. Probability Structures based on Atomless Probability Spaces
3. L^p-Banach lattices
4. Richly branching $\mathbb{R}$-trees

Stability in Continuous Logic

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As in classical logic, there are many (equivalent) ways of defining what it means for the complete continuous theory T to be *stable*:

- λ-stable for some λ;
- Existence of a *stable independence relation*
- Types over models are definable

All four of the theories described on the previous slide are stable. In fact, the first three are ω-stable and the last one is κ-stable if and only if $\kappa^\omega = \kappa$.

Simplicity in continuous logic

One can define *dividing* and *simplicity* in continuous logic exactly as it was defined in classical logic:

- A type $p_B(x) \in S_x(B)$ *does not divide* over A if whenever I is an A-indiscernible sequence with $B \in I$, then $\{p_{B'}(x) \mid B' \in I\}$ is consistent.
- T is *simple* if the relation $\smile\!\!\!\!\mid$ of dividing independence satisfies local character.

Do There Exist Simple Continuous Theories?

- All known examples of "essentially continuous" theories are either stable or not simple.
- Attempts to create essentially continuous simple, unstable theories failed, e.g. adding a generic predicate, applying the Keisler randomization procedure...

Question of Ben-Yaacov

Do there exist any "essentially continuous" simple, unstable theories?

Continuous Logic

Rosiness

The Urysohn Sphere

Defining $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$

T-classical complete theory, $\mathcal{M}$ a monster model for T.

$A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B \Leftrightarrow \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$.
Satisfies all axioms for a strict independence relation except base monotonicity.

$A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B \Leftrightarrow$ for all C' such that $C \subseteq C' \subseteq \operatorname{acl}(BC)$, we have $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C'} B$.
Satisfies all axioms for a strict independence relation except local character and extension.

$A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B \Leftrightarrow$ for all $B' \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B'$.

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Rosy Theories

Theorem (Adler)

$\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is a strict independence relation if and only if $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ has local character if and only if there is a strict independence relation for T at all. In this case, $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is the weakest strict independence relation for T, that is, if $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is another strict independence relation for T, then for all small A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} B$.

Definition

T is *rosy* if and only if $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ is a strict independence relation for T^{eq}.

Example

Simple theories and o-minimal theories are rosy.

Algebraic Closure in Continuous Logic

Suppose now $\mathcal{M}$ is a monster model for the complete continuous theory T.

Suppose $a \in \mathcal{M}$ and $B \subseteq \mathcal{M}$ is small. Then a is *algebraic over B* if the set of B-conjugates of a is a (metrically) *compact* subset of $\mathcal{M}$. (Equivalently, a lies in a compact B-definable subset of $\mathcal{M}$.)

In continuous logic, if $a \in \text{acl}(B)$, then there need not be a *finite* $B_0 \subseteq B$ such that $a \in \text{acl}(B_0)$. However, there will be a *countable* $B_0 \subseteq B$ such that $a \in \text{acl}(B_0)$; this is because definable sets in continuous logic may need countably many parameters for their definition.

Strict Countable Independence Relations

Definition

$\mathop{\smile\!\!\!\!\!\!|}^{*}$ is a *strict countable independence relation* if it satisfies all of the axioms for a strict independence relation except that it satisfies *countable character* instead of finite character, that is,

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{*} B \Leftrightarrow A_0 \mathop{\smile\!\!\!\!\!\!|}_{C}^{*} B \text{ for all countable } A_0 \subseteq A.$$

Theorem

Suppose that T is a complete continuous theory. Then $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ is a strict countable independence relation if and only if $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ has local character if and only if there is a strict countable independence relation for T at all. In this case, $\mathop{\smile\!\!\!\!\!\!|}^{\flat}$ is the weakest strict countable independence relation for T.

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Rosy Continuous Theories

We say that a continuous theory T is *rosy* if $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is a strict countable independence relation for T^{eq}. (We will say later what T^{eq} is for continuous logic.)

By the previous theorem, simple continuous theories are rosy.

In the rest of this talk, we aim to show that the theory of the *Urysohn sphere*, which is not simple, is rosy (with respect to finitary imaginaries).

Continuous Logic

Rosiness

The Urysohn Sphere

The Urysohn sphere

Recall that a *Polish metric space* is a complete, separable metric space.

The **Urysohn sphere** $\mathfrak{U}$ is the unique (up to isometry) Polish metric space of diameter ≤ 1 which is *universal* (all Polish metric spaces of diameter ≤ 1 isometrically embed in $\mathfrak{U}$) and *ultrahomogeneous* (any isometry between finite subsets of $\mathfrak{U}$ extends to an isometry of $\mathfrak{U}$).

L-the empty metric signature (consists solely of the metric symbol d, $d \leq 1$)
$T_{\mathfrak{U}}$-the L-theory of $\mathfrak{U}$
$\mathbb{U}$-a monster model for $T_{\mathfrak{U}}$

Model Theoretic Properties of $T_{\mathfrak{U}}$

Theorem (Henson)

1. *$T_{\mathfrak{U}}$ is $\aleph_0$-categorical;*
2. *$T_{\mathfrak{U}}$ admits QE;*
3. *$T_{\mathfrak{U}}$ is the model completion of the empty L-theory (so is the theory of existentially closed metric spaces of diameter ≤ 1);*
4. *for all $A \subseteq \mathbb{U}$, we have $\mathrm{acl}(A) = \overline{A}$.*

Thus, there appears to be an analogy between the theory of the Urysohn sphere in continuous logic and the theory of the infinite set in classical logic. However,...

$T_{\mathfrak{U}}$ is not simple

Theorem (Pillay)

$T_{\mathfrak{U}}$ is not simple.

Sketch.

- Let $A \subseteq \mathfrak{U}$ be small with all elements mutually $\frac{1}{2}$-apart. By QE, there is a unique type $p(x)$ determined by the conditions $\{d(x, a) = \frac{1}{4} \mid a \in A\}$.
- Let $B \subsetneq A$ be closed. We show that p divides over B, showing that $\mathop{\smile\!\!\!\!\!\!|}$ doesn't satisfy local character in $T_{\mathfrak{U}}$.
- Let $a \in A \setminus B$. We can find a B-indiscernible sequence $(a_i \mid i < \omega)$ of realizations of $\mathrm{tp}(a/B)$ which are mutually 1-apart. Then $d(x, a) = \frac{1}{4}$ 2-divides over B.

□

$T_{\mathfrak{U}}$ is real rosy

Theorem (Ealy, G.)

$T_{\mathfrak{U}}$ is real rosy, that is, $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ satisfies local character when restricted to the real sort.

Sketch.

1. By the triviality of acl in $T_{\mathfrak{U}}$, one can show that

$$A \mathop{\smile\!\!\!\!\!\!|}_{C}^{M} B \Leftrightarrow \overline{A} \cap \overline{B} \subseteq \overline{C}.$$

2. Next, show that $\mathop{\smile\!\!\!\!\!\!|}^{M} = \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ in $T_{\mathfrak{U}}$.
3. Suppose $A, B \subseteq \mathbb{U}$ are small. For $x \in \overline{A} \cap \overline{B}$, let $B_x \subseteq B$ be countable such that $x \in \overline{B_x}$. Let $B_0 := \bigcup\{B_x \mid x \in \overline{A} \cap \overline{B}\}$. Then $A \mathop{\smile\!\!\!\!\!\!|}_{B_0}^{\text{þ}} B$ and $|B_0| \leq \aleph_0 \cdot |\overline{A}|$, showing that $\mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}$ satisfies local character.

Why $\mathop{\smile\!\!\!\!\!\!|}\nolimits^{M} = \mathop{\smile\!\!\!\!\!\!|}\nolimits^{\text{þ}}$ in $T_{\mathfrak{U}}$

- It suffices to show that for any small, closed $A, B, C \subseteq \mathbb{U}$, there exists $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}\nolimits^{M}_{C} B$.
- Let $(a_i \mid i \in I)$ enumerate $A \setminus C$ and $(b_j \mid j \in J)$ enumerate $B \setminus C$.
- Let $\epsilon_i := d(a_i, C)$ and $\delta_{ij} := \max\{\epsilon_i, d(a_i, b_j)\}$.
- Let $\Sigma(X) := \text{tp}(A/C) \cup \{|d(x_i, b_j) - \delta_{i,j}| = 0 \mid i \in I, j \in J\}$. It suffices to show that Σ is satisfiable.
- To show that Σ is satisfiable, it suffices to show that Σ prescribes a metric on $X \cup B \cup C$.
- Check that all of the various triangle inequalities hold. This follows from the choice of δ_{ij}.

An Application of Real Rosiness

By the universality of $\mathfrak{U}$, we know that $\mathfrak{U}^n$ isometrically emdeds in $\mathfrak{U}$ for any $n \geq 2$. However,

Corollary

For any $n \geq 2$, there is no definable isometric embedding $\mathfrak{U}^n \to \mathfrak{U}$.

Proof.

First show that any definable isometric embedding $\mathfrak{U}^n \to \mathfrak{U}$ extends to an isometric embedding $\mathbb{U}^n \to \mathbb{U}$. (This actually takes work in continuous logic!) Then show that $U^{\text{þ}}_{\text{real}}(\mathbb{U}^n) = n$ and use monotonicity of $U^{\text{þ}}_{\text{real}}$-rank with respect to definable injections. □

One can also show that, for any $n \geq 2$, there is no A-definable injection $\mathfrak{U}^n \to \mathfrak{U}$, where $A \subseteq \mathfrak{U}$ is *finite*.

Definable Predicates

- Many issues around definability in continuous logic revolve around the notion of a definable predicate.
- Suppose, for each $n \in \mathbb{N}$, $\varphi_n(x, y_n)$ is a formula, where the y_n's are increasing finite tuples of variables. Then we obtain a definable predicate $P(x, Y)$ by taking the *forced limit* of the sequence $(\varphi_n(x, y_n))$.
- It should be viewed as a "formula" with finitely many object variables x and countably many parameter variables $Y := \bigcup_n y_n$.
- If $(\varphi_n(x, y_n))$ is a "fast" Cauchy sequence, then $P(x, y_n) = \lim \varphi_n(x, y_n)$.
- A predicate $P : \mathcal{M}^n \to [0, 1]$ is definable if and only if the map $\mathrm{tp}(a) \mapsto P(a) : S_n(T) \to [0, 1]$ is continuous.

$\mathcal{M}^{\text{eq}}$ in continuous logic

- As in classical logic, the eq-construction can be viewed as adding *canonical parameters* for formulae (or definable predicates in our case).
- Suppose $P(x, Y)$ is a definable predicate. On $\mathcal{M}_Y$, define the pseudometric $d_P(B, B') := \sup_x |P(x, B) - P(x, B')|$.
- In $\mathcal{M}^{\text{eq}}$, we add a sort $\mathcal{M}_P$, which is the metric space $\mathcal{M}_Y/(d_P = 0)$, as well as relevant "projection maps."
- The elements of $\mathcal{M}_P$ are canonical parameters of instances of $P(x, Y)$.
- If $|Y| < \omega$, we say that $P(x, Y)$ is a *finitary definable predicate* and, if $P(x, Y)$ is a finitary definable predicate, then the elements of $\mathcal{M}_P$ are called *finitary imaginaries*.
- $\mathcal{M}^{\text{feq}}$ is the reduct of $\mathcal{M}^{\text{eq}}$ where one only considers finitary imaginaries.

WEFI

Definition

We say that T has *weak elimination of finitary imaginaries*, abbreviated *WEFI*, if for every $e \in \mathcal{M}^{\text{feq}}$, there is a finite tuple $I(e)$ from $\mathcal{M}$ such that $e \in \text{dcl}(I(e))$ and $I(e) \in \text{acl}(e)$.

Equivalently, T has WEFI if and only if for every finitary definable predicate $\varphi(x)$, there is a finite tuple c from $\mathcal{M}$ such that $\varphi(x)$ is definable over c and whenever $\varphi(x)$ is defined over a finite tuple d, then $c \in \text{acl}(d)$.

A Fact About Iso($\mathbb{U}$)

We will need the following fact in our proof that $T_{\mathfrak{U}}$ has WEFI.

Theorem (J. Melleray)

Let A and B be finite subsets of $\mathbb{U}$. Set $G := \mathrm{Iso}(\mathbb{U}|A \cap B)$ and $H :=$ the subgroup of G generated by $\mathrm{Iso}(\mathbb{U}|A) \cup \mathrm{Iso}(\mathbb{U}|B)$. Then H is dense in G with respect to the topology of pointwise convergence.

$T_{\mathfrak{U}}$ has WEFI

- Suppose that $\varphi(x, a)$ is a finitary definable predicate.
- Let b be a subtuple of a such that $\varphi(x)$ is definable over b and $\varphi(x)$ is not definable over any proper subtuple of b.
- Now suppose that $\varphi(x)$ is definable over the finite tuple d. Let $c \in \mathbb{U}$. Let $G := \mathrm{Iso}(\mathbb{U}|b \cap d)$ and let H be the subgroup of G generated by $\mathrm{Iso}(\mathbb{U}|b) \cup \mathrm{Aut}(\mathbb{U}|d)$.
- If $\tau \in H$, then $\varphi(\tau(c)) = \varphi(c)$.
- If $\tau \in G$, then by the above theorem, there is a sequence (τ_n) from H such that $\tau_n(c) \to \tau(c)$.
- Since φ is continuous, we have

$$\varphi(\tau(c)) = \varphi(\lim \tau_n(c)) = \lim \varphi(\tau_n(c)) = \varphi(c).$$

- Thus, φ is defined over $b \cap d$.
- By choice of b, we have $b \cap d = b$, i.e. $b \in \mathrm{acl}(d)$.

Thus, we have that $T_{\mathfrak{U}}$ has WEFI.

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Thus, we have that $T_{\mathfrak{U}}$ has WEFI.

$T_{\mathfrak{U}}$ has WEFI

- Suppose that $\varphi(x, a)$ is a finitary definable predicate.
- Let b be a subtuple of a such that $\varphi(x)$ is definable over b and $\varphi(x)$ is not definable over any proper subtuple of b.
- Now suppose that $\varphi(x)$ is definable over the finite tuple d. Let $c \in \mathbb{U}$. Let $G := \text{Iso}(\mathbb{U}|b \cap d)$ and let H be the subgroup of G generated by $\text{Iso}(\mathbb{U}|b) \cup \text{Aut}(\mathbb{U}|d)$.
- If $\tau \in H$, then $\varphi(\tau(c)) = \varphi(c)$.
- If $\tau \in G$, then by the above theorem, there is a sequence (τ_n) from H such that $\tau_n(c) \to \tau(c)$.
- Since φ is continuous, we have

$$\varphi(\tau(c)) = \varphi(\lim \tau_n(c)) = \lim \varphi(\tau_n(c)) = \varphi(c).$$

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Thus, we have that $T_{\mathfrak{U}}$ has WEFI.

Real Rosy + WEFI $\Rightarrow$ Rosy w.r.t. $\mathcal{M}^{\text{feq}}$

In order to show that $T_{\mathfrak{U}}$ is rosy with respect to finitary imaginaries, it remains to prove

Theorem (Ealy, G.)

If T is real rosy and has WEFI, then T is rosy w.r.t. finitary imaginaries.

Outline of the Proof

Let us outline the proof of the above theorem. Suppose $A \subseteq \mathcal{M}^{\text{feq}}$ is small. We need to find a small cardinal κ such that for all small $D \subseteq \mathcal{M}^{feq}$, there is $C \subseteq D$ with $|C| < \kappa$ and such that $A \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{C} D$. Fix such a D.

- Let B be a small set of real elements whose image under the canonical maps equals A. Let κ witness local character for B (exists by real rosiness). We show that this is the desired κ.
- By choice of κ, there is $E \subseteq I(D)$ with $|E| < \kappa$ and $B \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{E} I(D)$.
- Let $C \subseteq D$ be such that $|C| < \kappa$ and such that $E \subseteq I(C)$. By base monotonicity, we have $B \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{I(C)} I(D)$.
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- Let B be a small set of real elements whose image under the canonical maps equals A. Let κ witness local character for B (exists by real rosiness). We show that this is the desired κ.
- By choice of κ, there is $E \subseteq I(D)$ with $|E| < \kappa$ and $B \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_E I(D)$.
- Let $C \subseteq D$ be such that $|C| < \kappa$ and such that $E \subseteq I(C)$. By base monotonicity, we have $B \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{I(C)} I(D)$.
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- Let B be a small set of real elements whose image under the canonical maps equals A. Let κ witness local character for B (exists by real rosiness). We show that this is the desired κ.
- By choice of κ, there is $E \subseteq I(D)$ with $|E| < \kappa$ and $B \mathop{\smile\!\!\!\!\!\!|}^{þ}_{E} I(D)$.
- Let $C \subseteq D$ be such that $|C| < \kappa$ and such that $E \subseteq I(C)$. By base monotonicity, we have $B \mathop{\smile\!\!\!\!\!\!|}^{þ}_{I(C)} I(D)$.
- Show that $B \mathop{\smile\!\!\!\!\!\!|}^{þ}_{C} D$.
- Show that $A \mathop{\smile\!\!\!\!\!\!|}^{þ}_{C} D$.

Superrosiness

In classical logic, a rosy theory is said to be *superrosy* if any type does not þ-fork over a finite subset of its domain. In analogy with the definition of supersimplicity in continuous logic, we make the following definition:

Definition

Suppose T is a rosy continuous theory with monster model $\mathcal{M}$. We say that T is *superrosy* if for all $a \in \mathcal{M}^{\mathrm{eq}}$, all small $B \subseteq \mathcal{M}^{\mathrm{eq}}$, and all $\epsilon > 0$, there is $c \in \mathcal{M}^{\mathrm{eq}}$, in the same sort as a with $d(a, c) < \epsilon$, and a finite $B_0 \subseteq B$ such that $c \mathop{\smile\!\!\!\!\!\!|}^{\text{þ}}_{B_0} B$

$T_{\mathfrak{U}}$ is superrosy w.r.t. $\mathbb{U}^{\mathrm{feq}}$

Theorem (Ealy, G.)

$T_{\mathfrak{U}}$ is superrosy with respect to finitary imaginaries.

Sketch of Proof

- First fix $a = (a_1, \ldots, a_n)$ a finite tuple from $\mathbb{U}$, $B \subseteq \mathbb{U}$ small, and $\epsilon > 0$.
- For each $i \in \{1, \ldots, n\}$, set $c_i := a_i$ if $a_i \notin \mathrm{acl}(B)$. Otherwise, set c_i to be an element of B within ϵ of a_i.
- Set $B_0 := \{c_1, \ldots, c_n\} \cap B$. Then $c \mathop{\smile\!\!\!\!\!\!\,|}^{\text{þ}}_{B_0} B$, whence $T_{\mathfrak{U}}$ is real superrosy.

The Proof Continued

- Now suppose $a \in \mathbb{U}^{\text{feq}}$, $B \subseteq \mathbb{U}^{\text{feq}}$ is small, and $\epsilon > 0$.
- Let a' be a representative of the equivalence class a. Choose $\delta > 0$ so that whenever c' is a tuple from $\mathbb{U}$ of the same kind as a' which is within δ of a', then $d(a, c) < \epsilon$, where c is the equivalence class of c'.
- By real superrosiness, we can find a finite tuple c' from $\mathbb{U}$ of the same kind as a' within δ of a' and such that $c' \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{C} I(B)$ for some finite $C \subseteq I(B)$.
- By base monotonicity, we may assume that $C = I(B_0)$ for some finite $B_0 \subseteq B$.
- Then, by earlier arguments, $c \mathop{\smile\!\!\!\!\!|}^{\text{þ}}_{B_0} B$, where c is the equivalence class of c'. $\square$

Questions

Question 1

Recall that we showed that Real Rosy +WEFI $\Rightarrow$ Rosy w.r.t. $\mathcal{M}^{\mathrm{feq}}$. This proof shows that, in classical logic, Real Rosy + WEI $\Rightarrow$ Rosy. Does this have any applications in the classical setting?

Question 2

Is $T_{\mathfrak{U}}$ rosy? What does $T_{\mathfrak{U}}^{\mathrm{eq}}$ look like? Does $T_{\mathfrak{U}}$ (weakly) eliminate hyperimaginaries?

Question 3

It is known that if T is a classical theory,

$$T \text{ simple, unstable} \Rightarrow T^R \text{ not simple,}$$

where T^R stands for the *Keisler randomization of* T. Is it true that T rosy implies T^R rosy? This would require having a better understanding of acl in T^R.