The Urysohn space is rosy

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The Birth of Stability Theory

Theorem (Morley, 1962)

If T is a theory in a countable language and is κ-categorical for some $\kappa > \aleph_0$, then T is λ-categorical for all $\lambda > \aleph_0$. T is then called uncountably categorical.

The techniques used to prove this theorem marked the beginning of *stability theory*: total transcendentality (AKA ω-stability), (Morley) ranks, etc…

Classification Theory

Theorem (Baldwin-Lachlin)

If T is an uncountably categorical theory, then T has either 1 countable model or $\aleph_0$ many countable models.

Theorem (Shelah-1970)

If T is κ-categorical for some $\kappa > |T|$, then T is λ-categorical for all $\lambda > |T|$. (Morley's theorem for uncountable languages.)

Theorem (Shelah)

If T is unstable, then T has 2^λ models of cardinality λ for $\lambda > |T|$.

Stable theories

Definition

T is *κ-stable* if for every $\mathcal{M} \models T$ and every $A \subseteq M$ with $|A| \leq \kappa$, we have $|S_1(A)| \leq \kappa$. T is said to be *stable* if it is κ-stable for some κ.

Example

The theory of the infinite set is ω-stable. Indeed, for each $a \in A$, there is a type determined by saying "$x = a$". There is also a type determined by saying "$x \neq a$" for each $a \in A$. Thus, there are $|A| + 1$-many 1-types over A.

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Stable theories (cont'd)

Example

Suppose that $T = ACF$. Suppose $K \models ACF$ and $A \subseteq K$. Without loss of generality, we may assume that $A = k$ is a subfield of K. Given $p \in S_1(k)$, define $I_p := \{f(x) \in k[x] \ : \ \text{“}f(x) = 0\text{”} \in p\}$. Then $p \mapsto I_P$ is a bijection between $S_1(k)$ and the set of prime ideals in $k[x]$; the latter set has cardinality $|k| + \aleph_0$ since every ideal in $k[x]$ is finitely generated by Hilbert's basis theorem.

Example

DCF is ω-stable.

Unstable theories

Example

The theory of the random graph is *not* stable. Fix κ and let $G \models T_{rg}$ be κ^+-saturated. Then one can find κ many elements A that are not connected to each other. For $X \subseteq A$, let $p_X(x)$ be the type declaring xEa for $a \in X$ and $\neg xEa$ for $a \notin X$. Then these p_X's are distinct, so there are 2^κ many types over A.

Example

o-minimal theories are not stable.

Theorem

T is unstable if and only if there is $\mathcal{M} \models T$, a formula $\varphi(x, y)$, and sequences $(a_i), (b_i)$ from M such that $\mathcal{M} \models \varphi(a_i, b_j) \Leftrightarrow i < j$.

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Continuous Stable Theories

One can define κ-stability for continuous theories just as for classical (discrete theories). However, there is an alternate (metric) notion of κ-stability, and it is usually this notion that is referred to. Fortunately, they yield the same class of stable theories.

Examples

1. Infinite-dimensional Hilbert spaces (ω-stable)
2. Atomless probability algebras (ω-stable)
3. L^p-Banach lattice (ω-stable)
4. Richly branching $\mathbb{R}$-trees (κ-stable if and only if $\kappa^\omega = \kappa$)

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An application to functional analysis

Suppose that M is some separable object of functional analysis (e.g C^*-algebra, von-Neumann algebra, etc...) and $\mathcal{U}$, $\mathcal{V}$ are nonprincipal ultrafilters on $\mathbb{N}$. Is it true that $\mathcal{M}^{\mathcal{U}} \cong \mathcal{M}^{\mathcal{V}}$? Under (CH), the answer is yes. But what about under $\neg$(CH).

Theorem (Hart, Farah, Sherman)

Suppose that $\neg$(CH) holds. Suppose that $\mathcal{M}$ is a separable metric structure.

1. *If $\mathrm{Th}(\mathcal{M})$ is stable, then all nonprincipal ultrapowers of $\mathcal{M}$ over $\mathbb{N}$ are isomorphic.*
2. *If $\mathrm{Th}(\mathcal{M})$ is unstable, then there are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ such that $\mathcal{M}^{\mathcal{U}} \not\cong \mathcal{M}^{\mathcal{V}}$.*

II_1-factors are unstable as are unital C^*-algebras and their unitary groups.

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$T_{\mathfrak{U}}$ is not stable

Remember that $\mathfrak{U}$ denotes the Urysohn sphere and $T_{\mathfrak{U}}$ is the theory of $\mathfrak{U}$. In this talk, $\mathbb{U}$ is a very saturated model of $T_{\mathfrak{U}}$. Fix a (small) cardinal λ and let A be a set of elements of $\mathbb{U}$ of size $< \lambda$ which are pairwise distance 1 apart. Then for any $X \subseteq \lambda$, the collection of conditions

$$\Gamma_X := \{d(x, a_i) = 1 \mid i \in X\} \cup \{d(x, a_i) = \frac{1}{2} \mid i \notin X\}$$

is finitely satisfiable in $\mathbb{U}$. This yields 2^{λ} many distinct complete 1-types over A.

Free extensions

Fix a theory T and a "*monster model*" $\mathfrak{M} \models T$ (a very saturated and homogeneous model). Throughout, $A, B, C \subseteq \mathfrak{M}$ are *small* in the sense that they have cardinality less than the saturation level of $\mathfrak{M}$. A *model* will refer to a small elementary substructure of $\mathfrak{M}$.

- Stable theories have a nice notion of *independence* for small subsets of $\mathfrak{M}$.
- The idea is *A is independent from B over C*, written $A \mathop{\smile\!\!\!\!\!\!|}_C B$, if $B \cup C$ gives no more information about A than C does.
- In terms of types, we say that $\text{tp}(A/BC)$ is a *free* or *nonforking* extension of $\text{tp}(A/C)$.
- A very important property of this independence notion is that of *extension*, namely that if $p(x)$ is a type over C and $B \supseteq C$, then p *has* a free extension to B.

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Definable types

Definition

A type $p(x) \in S(A)$ is *definable* if for every formula $\varphi(x, y)$ without parameters, there is another formula $d_p\varphi(y)$ with parameters from A such that, for every $a \in A$, $\varphi(x, a) \in p \Leftrightarrow \models d_p\varphi(a)$.

Theorem

T is stable if and only if every type over a model is definable.

Suppose that T is stable and $p(x) \in S(M)$, where M is a model. If $M \subseteq B \subseteq \mathfrak{M}$, then define $q(x) = \{\varphi(x, b) \ : \models d_p\varphi(b), b \in B\}$. Then one can show that $q(x) \in S(B)$. In this case, $q(x)$ is a free extension of $p(x)$. In fact, it is the *unique* free extension of $p(x)$ to B.

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Stable Independence Relations

Suppose that T is stable. Then $\mathop{\smile\!\!\!\!\!\mid}$ satisfies the following properties:

1. Automorphism invariance
2. Symmetry: $A \mathop{\smile\!\!\!\!\!\mid}_C B \Leftrightarrow B \mathop{\smile\!\!\!\!\!\mid}_C A$
3. Transitivity: $A \mathop{\smile\!\!\!\!\!\mid}_C BD \Leftrightarrow A \mathop{\smile\!\!\!\!\!\mid}_C B$ and $A \mathop{\smile\!\!\!\!\!\mid}_{BC} D$
4. Finite character: $A \mathop{\smile\!\!\!\!\!\mid}_C B$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}_C B$ for all finite tuples a from A
5. Extension: for all A, B, C, there exists $A' \models \mathrm{tp}(A/C)$ such that $A' \mathop{\smile\!\!\!\!\!\mid}_C B$
6. Local Character: If a is any finite tuple, then there is $B_0 \subseteq B$ of cardinality $\leq |T|$ such that $a \mathop{\smile\!\!\!\!\!\mid}_{B_0} B$
7. Stationarity of Types: If $\mathrm{tp}(A/M) = \mathrm{tp}(A'/M)$, $A \mathop{\smile\!\!\!\!\!\mid}_M B$, and $A' \mathop{\smile\!\!\!\!\!\mid}_M B$, then $\mathrm{tp}(A/MB) = \mathrm{tp}(A'/MB)$.

Stable Independence Relations (cont'd)

Definition

Any relation $\mathop{\smile\!\!\!\!\!\!|}^*$ that satifies the properties (1)-(7) is called a *stable independence relation*.

Theorem

1. *If T is stable, then there is a unique stable independence relation, namely nonforking independence.*
2. *If T admits a stable independence relation, then T is stable (and this stable independence relation must be nonforking independence).*

Forking in ACF

- In ACF, there is a nice geometric interpretation of $\downarrow$.
- Suppose that $K \models ACF$ and $k \subseteq l \subseteq K$ are subfields.
- For $a \in K$, define $RM(a/k) := d$ if a is the generic point of an irreducible variety V defined over k of dimension d.
- Then $a \downarrow_k l$ if and only if $RM(a/k) = RM(a/l)$.

A combinatorial approach to forking

Definition

Suppose that $\varphi(x, a)$ is a formula and $A \subseteq \mathfrak{M}$ is small.

1. $\varphi(x, a)$ *divides over* A if there is an A-indiscernible sequence $(a_i \mid i < \omega)$ with $\mathrm{tp}(a/A) = \mathrm{tp}(a_0/A)$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.
2. $\varphi(x, a)$ *forks over* A if there are $\varphi_1(x), \ldots, \varphi_n(x)$, each of which divide over A, such that $\models \varphi(x) \rightarrow \bigvee_{i=1}^{n} \varphi_i(x)$.

- "Forking=negligible or smaller dimension"
- If $p(x) \in S(B)$ and $A \subseteq B$, then p forks over A if it contains a formula that forks over A. So *nonforking* extensions don't include any "lower-dimensional" sets which provide more information about realizations of p than A.

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Simple theories

Definition

T is *simple* if $\downarrow$ satisfies local character.

Example

The theory of the random graph, which is not stable, is simple.

If T is simple, then $\downarrow$ satisfies the first six properties of a stable independence relation but stationarity of types might fail. A useful substitute is:

Theorem (Independence Theorem)

Suppose that T is simple, M is a model, and $A, B \supseteq M$ are such that $A \downarrow_M B$. If $p(x) \in S(A)$ and $q(x) \in S(B)$ are nonforking extensions of p_0, their restriction to M, then $p \cup q$ is consistent and is a nonforking extension of p_0. (Type Amalgamation over Models)

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Characterizing simple theories

Call $\mathop{\smile\!\!\!\!\!\!|}^*$ a *simple independence relation* if $\mathop{\smile\!\!\!\!\!\!|}^*$ satisfies 1-6 and the Independence Theorem.

Theorem

If T is simple, then $\mathop{\smile\!\!\!\!\!\!|}$ is the unique simple independence relation. If T has a simple independence relation, then T is simple.

Example

For G a big model of the theory of the random graph, define $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ if and only if $A \cap B \subseteq C$. Then $\mathop{\smile\!\!\!\!\!\!|}^*$ is a simple independence relation and thus $\mathop{\smile\!\!\!\!\!\!|}^*$ is the relation of nonforking independence.

$T_{\mathfrak{U}}$ is not simple

Since $T_{\mathfrak{U}}$ contains a copy of the random graph inside, maybe it is simple.

Theorem (Pillay)

$T_{\mathfrak{U}}$ is not simple.

Sketch.

- Let $A \subseteq \mathbb{U}$ be small with all elements mutually $\frac{1}{2}$-apart. By QE, there is a unique type $p(x)$ determined by the conditions $\{d(x,a) = \frac{1}{4} \mid a \in A\}$.
- Let $B \subsetneq A$ be closed. We show that p divides over B, showing that $\downarrow$ doesn't satisfy local character in $T_{\mathfrak{U}}$.
- Let $a \in A \setminus B$. We can find a B-indiscernible sequence $(a_i \mid i < \omega)$ of realizations of $\text{tp}(a/B)$ which are mutually 1-apart. Then "$d(x,a) = \frac{1}{4}$" 2-divides over B.

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DLO

Example

DLO is not simple. To see this, fix $b < a < c$. Then $bc \mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} a$: $\mathrm{tp}(bc/a)$ is determined by the formula $y < a < z$ which doesn't divide over $\emptyset$. However, $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{\emptyset} bc$. Look at the indiscernible sequence

$$b = b_0 < c = c_0 < b_1 < c_1 < b_2 < c_2 \cdots$$

Then if $\varphi(x, y, z)$ is the formula $y < x < z$, then $\{\varphi(x, b_i, c_i) \mid i < \omega\}$ is 2-inconsistent, so $\varphi(x, b, c)$ divides over $\emptyset$ and is in $\mathrm{tp}(a/bc)$.

More generally, any o-minimal theory is not simple.

Independence in o-minimal theories

Suppose that T is o-minimal.

Definition

If $X \subseteq \mathfrak{M}^n$ is definable, then $\dim(X)$ is the dimension of the biggest open cell contained in X. If $a \in \mathfrak{M}^n$ and $A \subseteq \mathfrak{M}$, we define $\dim(a/A) := \min\{\dim(X) \mid X \text{ is A-definable and } a \in X\}$.

Define $a \mathop{\smile\!\!\!\!\!\!\mid}^{o}_{C} B$ if and only if $\dim(a/BC) = \dim(a/C)$. Then $\mathop{\smile\!\!\!\!\!\!\mid}^{o}$ is a very well-behaved independence relation and one can use it in many ways to mimic arguments from stability and simplicity theory.

Question

Is there a common framework for simple theories and o-minimal theories?

Defining $\mathop{\smile\!\!\!\!\!\!|}^{þ}$

T-classical complete theory, $\mathcal{M}$ a monster model for T.

$A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C} B \Leftrightarrow \operatorname{acl}(AC) \cap \operatorname{acl}(BC) = \operatorname{acl}(C)$.

Satisfies all axioms for a strict independence relation except perhaps *base monotonicity*: If $D \subseteq C \subseteq B$ and $A \mathop{\smile\!\!\!\!\!\!|}_{D} B$, then $A \mathop{\smile\!\!\!\!\!\!|}_{C} B$.

$A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B \Leftrightarrow$ for all C' such that $C \subseteq C' \subseteq \operatorname{acl}(BC)$, we have $A \mathop{\smile\!\!\!\!\!\!|}^{a}_{C'} B$. Satisfies all axioms for a strict independence relation except perhaps local character and extension.

$A \mathop{\smile\!\!\!\!\!\!|}^{þ}_{C} B \Leftrightarrow$ for all $E \supseteq BC$ there is $A' \models \operatorname{tp}(A/BC)$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} E$.

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Rosy Theories

Theorem (Adler, Ealy, Onshuus)

$\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is a strict independence relation if and only if $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ has local character if and only if there is a strict independence relation for T at all. In this case, $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is the weakest strict independence relation for T, that is, if $\mathop{\smile\!\!\!\!\!\!|}^{*}$ is another strict independence relation for T, then for all small A, B, C, we have $A \mathop{\smile\!\!\!\!\!\!|}^{*}_{C} B \Rightarrow A \mathop{\smile\!\!\!\!\!\!|}^{þ}_{C} B$.

Definition

T is *rosy* if and only if $\mathop{\smile\!\!\!\!\!\!|}^{þ}$ is a strict independence relation for T^{eq}.

Example

Simple theories and o-minimal theories are rosy.

Strict Countable Independence Relations

Definition

$\mathop{\smile\hskip-0.9em ^{\mid}}^{*}$ is a *strict countable independence relation* if it satisfies all of the axioms for a strict independence relation except that it satisfies *countable character* instead of finite character, that is,

$$A \mathop{\smile\hskip-0.9em ^{\mid}}^{*}_{C} B \Leftrightarrow A_0 \mathop{\smile\hskip-0.9em ^{\mid}}^{*}_{C} B \text{ for all countable } A_0 \subseteq A.$$

Theorem

Suppose that T is a complete continuous theory. Then $\mathop{\smile\hskip-0.9em ^{\mid}}^{þ}$ is a strict countable independence relation if and only if $\mathop{\smile\hskip-0.9em ^{\mid}}^{þ}$ has local character if and only if there is a strict countable independence relation for T at all. In this case, $\mathop{\smile\hskip-0.9em ^{\mid}}^{þ}$ is the weakest strict countable independence relation for T.

$T_{\mathfrak{U}}$ is real rosy

Theorem (Ealy, G.)

$T_{\mathfrak{U}}$ is real rosy, that is, $\mathop{\smile\hspace{-0.9em}\mid}^{þ}$ satisfies local character when restricted to the real sort.

Sketch.

1. By the triviality of acl in $T_{\mathfrak{U}}$, one can show that

$$A \mathop{\smile\hspace{-0.9em}\mid}^{M}_{C} B \Leftrightarrow \overline{A} \cap \overline{B} \subseteq \overline{C}.$$

2. Next, show that $\mathop{\smile\hspace{-0.9em}\mid}^{M} = \mathop{\smile\hspace{-0.9em}\mid}^{þ}$ in $T_{\mathfrak{U}}$.
3. Suppose $A, B \subseteq \mathbb{U}$ are small. For $x \in \overline{A} \cap \overline{B}$, let $B_x \subseteq B$ be countable such that $x \in \overline{B_x}$. Let $B_0 := \bigcup\{B_x \mid x \in \overline{A} \cap \overline{B}\}$. Then $A \mathop{\smile\hspace{-0.9em}\mid}^{þ}_{B_0} B$ and $|B_0| \leq \aleph_0 \cdot |\overline{A}|$, showing that $\mathop{\smile\hspace{-0.9em}\mid}^{þ}$ satisfies local character.

$\mathop{\smile\!\!\!\!\!\!|}^{M} = \mathop{\smile\!\!\!\!\!\!|}^{þ}$ in $T_{\mathfrak{U}}$

- It suffices to show that for any small, closed $A, B, C \subseteq \mathbb{U}$, there exists $A' \equiv_C A$ with $A' \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B$.
- Let $(a_i \mid i \in I)$ enumerate $A \setminus C$ and $(b_j \mid j \in J)$ enumerate $B \setminus C$.
- Let $\epsilon_i := d(a_i, C)$ and $\delta_{ij} := \max\{\epsilon_i, d(a_i, b_j)\}$.
- Let $\Sigma(X) := \mathrm{tp}(A/C) \cup \{|d(x_i, b_j) - \delta_{i,j}| = 0 \mid i \in I, j \in J\}$. It suffices to show that Σ is satisfiable.
- To show that Σ is satisfiable, it suffices to show that Σ prescribes a metric on $X \cup B \cup C$.
- Check that all of the various triangle inequalities hold. This follows from the choice of δ_{ij}.

An Application of Real Rosiness

By the universality of $\mathfrak{U}$, we know that $\mathfrak{U}^n$ isometrically emdeds in $\mathfrak{U}$ for any $n \geq 2$. However,

Corollary

For any $n \geq 2$, there is no definable isometric embedding $\mathfrak{U}^n \to \mathfrak{U}$.

Proof.

First show that any definable isometric embedding $\mathfrak{U}^n \to \mathfrak{U}$ extends to an isometric embedding $\mathbb{U}^n \to \mathbb{U}$. (Recall that this actually takes work in continuous logic!) Then show that $U^{\text{þ}}_{\text{real}}(\mathbb{U}^n) = n$ and use monotonicity of $U^{\text{þ}}_{\text{real}}$-rank with respect to definable injections. □

Definable Predicates

- Many issues around definability in continuous logic revolve around the notion of a definable predicate.
- Suppose, for each $n \in \mathbb{N}$, $\varphi_n(x, y_n)$ is a formula, where the y_n's are increasing finite tuples of variables. Suppose also that $u : [0,1]^{\mathbb{N}} \to [0,1]$ is a continuous function. Then we obtain a definable predicate $P(x, Y) := u((\varphi_n(x, y_n)))$, where $Y := \bigcup_n y_n$.
- It should be viewed as a "formula" with finitely many object variables x and countably many parameter variables Y.

$\mathcal{M}^{eq}$ in continuous logic

- As in classical logic, the eq-construction can be viewed as adding *canonical parameters* for formulae (or definable predicates in our case).
- Suppose $P(x, Y)$ is a definable predicate. On $\mathcal{M}_Y$, define the pseudometric $d_P(B, B') := \sup_x |P(x, B) - P(x, B')|$.
- In $\mathcal{M}^{eq}$, we add a sort $\mathcal{M}_P$, which is the metric space $\mathcal{M}_Y/(d_P = 0)$, as well as relevant "projection maps."
- The elements of $\mathcal{M}_P$ are canonical parameters of instances of $P(x, Y)$, meaning an automorphism preserves $P(x, B)$ if and only if it fixes the equivalence class of B.
- If $|Y| < \omega$, we say that $P(x, Y)$ is a *finitary definable predicate* and. If $P(x, Y)$ is a finitary definable predicate, then the elements of $\mathcal{M}_P$ are called *finitary imaginaries*.
- $\mathcal{M}^{feq}$ is the reduct of $\mathcal{M}^{eq}$ where one only considers finitary imaginaries.

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WEFI

Definition

We say that T has *weak elimination of finitary imaginaries* (*WEFI*) if for every finitary definable predicate $\varphi(x)$, there is a finite tuple c from $\mathcal{M}$ such that $\varphi(x)$ is definable over c and whenever $\varphi(x)$ is defined over a finite tuple d, then $c \in \mathrm{acl}(d)$.

Equivalently, for every $e \in \mathcal{M}^{\mathrm{feq}}$, there is a finite tuple $I(e)$ from $\mathcal{M}$ such that $e \in \mathrm{dcl}(I(e))$ and $I(e) \in \mathrm{acl}(e)$. ($I(e)$ is a "weak code" for e.)

A Fact About Iso($\mathbb{U}$)

We needed the following fact in our proof that $T_{\mathfrak{U}}$ has WEFI.

Theorem (J. Melleray)

Let A and B be finite subsets of $\mathbb{U}$. Set $G := \mathrm{Iso}(\mathbb{U}|A \cap B)$ and $H :=$ the subgroup of G generated by $\mathrm{Iso}(\mathbb{U}|A) \cup \mathrm{Iso}(\mathbb{U}|B)$. Then H is dense in G with respect to the topology of pointwise convergence.

$T_{\mathfrak{U}}$ has WEFI

- Suppose that $\varphi(x, a)$ is a finitary definable predicate.
- Let b be a subtuple of a such that $\varphi(x)$ is definable over b and $\varphi(x)$ is not definable over any proper subtuple of b.
- Now suppose that $\varphi(x)$ is definable over the finite tuple d. Let $G := \text{Iso}(\mathbb{U}|b \cap d)$ and let H be the subgroup of G generated by $\text{Iso}(\mathbb{U}|b) \cup \text{Aut}(\mathbb{U}|d)$. Let $c \in \mathbb{U}$.
- If $\tau \in H$, then $\varphi(\tau(c)) = \varphi(c)$.
- If $\tau \in G$, then by the above theorem, there is a sequence (τ_n) from H such that $\tau_n(c) \to \tau(c)$.
- Since φ is continuous, we have

$$\varphi(\tau(c)) = \varphi(\lim \tau_n(c)) = \lim \varphi(\tau_n(c)) = \varphi(c).$$

- Thus, φ is defined over $b \cap d$.
- By choice of b, we have $b \cap d = b$, i.e. $b \in \text{acl}(d)$.

Thus, we have that $T_{\mathfrak{U}}$ has WEFI.

Real Rosy + WEFI $\Rightarrow$ Rosy w.r.t. $\mathcal{M}^{\text{feq}}$

Theorem (Ealy, G.)

If T is real rosy and has WEFI, then T is rosy w.r.t. finitary imaginaries.

Corollary

$T_{\mathfrak{U}}$ is rosy with respect to finitary imaginaries.

Questions

What about arbitrary imaginaries? Can we (weakly) eliminate them? Is $T_{\mathfrak{U}}$ rosy?

References

- H. Adler, *A geometric introduction to forking and thorn-forking*
- I. Ben Yaacov, A. Berenstein, C. W. Henson, A. Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis. Vol. 2, pgs. 315-427, London Math. Soc. Lecture Note Ser. (350), Cambridge Univ. Press, Cambridge, 2008.
- C. Ealy and I. Goldbring, *Thorn-forking in continuous logic*, To appear in the Journal of Symbolic Logic. Available at `http://www.math.ucla.edu/~isaac`
- A. Pillay, *An introduction to stability theory*