Definable Functions in Urysohn's Metric Space

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Metric Structures

- A (bounded) metric structure is a (bounded) complete metric space (M, d), together with distinguished
  1. elements,
  2. functions (mapping M^n into M for various n), and
  3. predicates (mapping M^n into a bounded interval in $\mathbb{R}$ for various n).
- Each function and predicate is required to be uniformly continuous.
- For the sake of simplicity, we suppose that the metric is bounded by 1 and the predicates all take values in $[0, 1]$.

Examples of Metric Structures

1. If $\mathcal{M}$ is a structure from classical model theory, then we can consider $\mathcal{M}$ as a metric structure by equipping it with the discrete metric. If $P \subseteq M^n$ is a distinguished predicate, then we consider it as a mapping $P : M^n \to \{0, 1\} \subseteq [0, 1]$ by

$$P(a) = 0 \text{ if and only if } \mathcal{M} \models P(a).$$

2. Suppose X is a Banach space with unit ball B. Then $(B, 0_X, \|\cdot\|, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure, where $f_{\alpha,\beta} : B^2 \to B$ is given by $f(x, y) = \alpha \cdot x + \beta \cdot y$ for all scalars α and β with $|\alpha| + |\beta| \leq 1$.
3. If H is a Hilbert space with unit ball B, then $(B, 0_H, \|\cdot\|, \langle\cdot,\cdot\rangle, (f_{\alpha,\beta})_{\alpha,\beta})$ is a metric structure.

Bounded Continuous Signatures

- As in classical logic, a signature L for continuous logic consists of constant symbols, function symbols, and predicate symbols, the latter two coming also with arities.
- New to continuous logic: For every function symbol F, the signature must specify a *modulus of uniform continuity* Δ_F, which is a function $\Delta_F : (0,1] \to (0,1]$. Likewise, a modulus of uniform continuity is specified for each predicate symbol.
- The metric d is included as a (logical) predicate in analogy with $=$ in classical logic.

L-structures

An *L-structure* is a metric structure $\mathcal{M}$ whose distinguished constants, functions, and predicates are interpretations of the corresponding symbols in L. Moreover, the uniform continuity of the functions and predicates is witnessed by the moduli of uniform continuity specified by L.

e.g. If P is a unary predicate symbol, then for all $\epsilon > 0$ and all $x, y \in M$, we have:

$$d(x, y) < \Delta_P(\epsilon) \Rightarrow |P^{\mathcal{M}}(x) - P^{\mathcal{M}}(y)| \leq \epsilon.$$

Formulae

- Terms are defined as in classical logic.
- Atomic formulae are of the form $d(t_1, t_2)$ and $P(t_1, \ldots, t_n)$ where P is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms.
- Connectives: If $\varphi_1, \ldots, \varphi_n$ are formulae and $u : [0,1]^n \to [0,1]$ is any continuous function, then $u(\varphi_1, \ldots, \varphi_n)$ is a formula.
- Quantifiers: If φ is a formula, then so is $\sup_x \varphi$ and $\inf_x \varphi$. (sup "$=$" $\forall$ and inf "$=$" $\exists$)
- If $\varphi(x_1, \ldots, x_n)$ is an L-formula, $\mathcal{M}$ an L-structure, and $a_1, \ldots, a_n$ elements of M, then $\mathcal{M}$ gives a value $\varphi^{\mathcal{M}}(a_1, \ldots, a_n)$, which is a number in $[0,1]$ measuring "how true" φ is when $a_1, \ldots, a_n$ are plugged in for the free variables.
- $t^{\mathcal{M}} : M^n \to M$ and $\varphi^{\mathcal{M}} : M^n \to [0,1]$ are uniformly continuous for any term t and any formula φ (with Δ_t and Δ_φ calculable from the moduli in the signature.)

Theories

- A *condition* is an expression of the form "$\varphi = 0$", where φ is a formula. If φ is a sentence, then the condition "$\varphi = 0$" is called a *closed condition*.

Example

In the signature for Hilbert spaces, the condition $\langle x, y\rangle = 0$ expresses that x and y are orthogonal. The closed condition

$$\inf_{x_1} \cdots \inf_{x_n} \max_{i,j} |\langle x_i, x_j\rangle - \delta_{ij}| = 0$$

expresses that, for any $\epsilon > 0$, there are $x_1, \ldots, x_n$ such that $\langle x_i, x_j\rangle < \epsilon$ and $|\|x_i\| - 1| < \epsilon$. In an ω_1*-saturated* structure, where inf's are realized, it will express that there are n mutually orthogonal unit vectors.

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Theories

- A *condition* is an expression of the form "$\varphi = 0$", where φ is a formula. If φ is a sentence, then the condition "$\varphi = 0$" is called a *closed condition*.
- We can express weak inequalities as conditions: $\varphi \leq \psi$ can be expressed as $\varphi \dot{-} \psi = 0$, where $a \dot{-} b = \max(0, a - b)$.
- An *L-theory* is a set of closed L-conditions.
- If $\mathcal{M}$ is an L-structure, then the *theory of* $\mathcal{M}$ is the theory

$$\mathrm{Th}(\mathcal{M}) := \{\text{"}\varphi = 0\text{"} \mid \varphi \text{ a sentence}, \varphi^{\mathcal{M}} = 0\}.$$

- If $\varphi^{\mathcal{M}} = r$, then $|\varphi^{\mathcal{M}} - r| = 0$, so "$|\varphi - r| = 0$" will be in the theory of $\mathcal{M}$.
- An L-theory is *complete* if it is of the form $\mathrm{Th}(\mathcal{M})$ for some L-structure $\mathcal{M}$.

Examples of Complete Continuous Theories

1. Infinite-dimensional Hilbert spaces (over $\mathbb{R}$)
2. Probability algebras based on atomless probability spaces
3. L^p-Banach lattices
4. Richly branching $\mathbb{R}$-trees

Definable and algebraic closure

Definition

Suppose that $\mathcal{M}$ is a structure and $A \subseteq M$. If $b \in M$, we say:

- $b \in \mathrm{dcl}(A)$ if $\{b\}$ is an A-definable set.
- $b \in \mathrm{acl}(A)$ if b lives in a *compact* A-definable set.

Saturated Structures

Definition

If $\mathcal{M}$ is an L-structure and $A \subseteq M$ is a parameterset, then a collection $p(x)$ of $L(A)$-conditions is a *(complete) type over A* if there is $\mathcal{M} \preceq \mathcal{N}$ and $b \in N^{|x|}$ such that $p(x) = \{\varphi(x) = 0 \ : \ \varphi^{\mathcal{N}}(b) = 0, \ \varphi(x) \in L(A)\}$.

Definition

If κ is an infinite cardinal, a structure $\mathcal{M}$ is said to be *κ-saturated* if every type over a parameterset of cardinality $< \kappa$ is realized in M.

Fact

Given any infinite cardinal κ and any structure $\mathcal{M}$, there is an elementary extension $\mathcal{M} \preceq \mathcal{N}$ such that $\mathcal{N}$ is κ-saturated.

Definable and algebraic closure-restated

Definition

Suppose that $\mathcal{M}$ is a ω_1*-saturated structure* and $A \subseteq M$. If $b \in M$, we say:

- $b \in \mathrm{dcl}(A)$ if $\sigma(b) = b$ for all $\sigma \in \mathrm{Aut}(\mathcal{M}/A)$.
- $b \in \mathrm{acl}(A)$ if the orbit of b under the action of $\mathrm{Aut}(\mathcal{M}/A)$ is compact.

It is clear from the above description that $\bar{A} \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ for all $A \subseteq M$, even if $\mathcal{M}$ is not saturated.

Remark

For my next talk, it will be relevant to note that dcl and acl have *countable character*: $b \in \mathrm{dcl}(A)$ if and only if $b \in \mathrm{dcl}(A_0)$ for some countable $A_0 \subseteq A$.

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The Urysohn Sphere

Definition

A *Polish metric space* is a separable, complete metric space.

Definition

The *Urysohn sphere* $\mathfrak{U}$ is the unique (up to isometry) Polish metric space of diameter 1 which is:

1. universal- all Polish metric spaces of diameter ≤ 1 admit an isometric embedding into $\mathfrak{U}$;
2. ultrahomogeneous- if $\phi : X_1 \to X_2$ is an isometry between finite subspaces of $\mathfrak{U}$, then there is an isometry $\tilde{\phi} : \mathfrak{U} \to \mathfrak{U}$ extending ϕ.

Existence: Urysohn, Katětov; alternatively, it is the Fraisse limit of finite metric spaces of diameter ≤ 1 (in the sense of continuous logic)

Axioms for the theory of $\mathfrak{U}$

- In this slide, a formula $\theta(x_1, \ldots, x_n)$ denotes a formula of the form $\max_{i,j} |d(x_i, x_j) - r_{ij}|$, where (r_{ij}) is a distance matrix for a finite metric space of diameter ≤ 1.
- Then for any such formula $\theta(x_1, \ldots, x_n, x_{n+1})$ and any $\epsilon > 0$, there is a $\delta > 0$ such that, for $a_1, \ldots, a_n \in \mathfrak{U}$ satisfying $(\theta \upharpoonright n)(a_1, \ldots, a_n) < \delta$, there exists $a_{n+1} \in \mathfrak{U}$ such that $\theta(a_1, \ldots, a_n, a_{n+1}) \leq \epsilon$.
- We let $T_{\mathfrak{U}}$ denote the set of axioms of the form:

$$\forall \vec{x} \exists y ((\theta \upharpoonright n)(\vec{x}) < \delta \rightarrow \theta(\vec{x}, y) \leq \epsilon).$$

More precisely,

$$\sup_{\vec{x}} \inf_y (\min \left(\frac{\epsilon}{1-\delta}(1 - (\theta \upharpoonright n)(\vec{x})), \theta(\vec{x}, y) \right) \dot{-} \epsilon) = 0.$$

Basic Model Theory of $T_{\mathfrak{U}}$

Theorem (Folklore/Henson/Usvyatsov)

1. *$T_{\mathfrak{U}}$ is $\aleph_0$-categorical, whence equal to $\mathrm{Th}(\mathfrak{U})$;*
2. *$T_{\mathfrak{U}}$ admits QE;*
3. *$T_{\mathfrak{U}}$ is the model completion of the empty L-theory (so is the theory of existentially closed metric spaces of diameter ≤ 1);*
4. *for all $A \subseteq \mathfrak{U}$, we have $\mathrm{acl}(A) = \overline{A}$, so dcl and acl are trivial.*

So $T_{\mathfrak{U}}$ is like a continuous analogue of the theory of the infinite set in classical logic. (And in other ways, it's drastically different!-See next talk.)

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Proof of Fact 4

Lemma

$\mathrm{acl}(A) = \bar{A}$.

Proof.

Work in an ω_1-saturated elementary extension $\mathbb{U}$ of $\mathfrak{U}$. Suppose that $b \notin \bar{A}$. Consider the following collection of formulae:

$$\{d(x_i, a) = d(b, a) \ : \ i < \omega, \ a \in A\} \cup \{d(x_i, x_j) = 2 \odot d(b, \bar{A}) \ : \ i < j < \omega\}.$$

Any finite subset defines a metric space, so can be realized in $\mathfrak{U}$. By ω_1-saturation, we can find $(b_i \ : \ i < \omega)$ in $\mathbb{U}$ realizing this partial type. By quantifier-elimination, $\mathrm{tp}(b_i/A) = \mathrm{tp}(b/A)$ for all $i < \omega$. But (b_i) has no convergent subsequence, so the orbit of b under $\mathrm{Aut}(\mathbb{U}/A)$ is not compact, whence $b \notin \mathrm{acl}(A)$. □

Definable predicates

For purposes of definability in continuous logic, formulae aren't expressive enough. It turns out that we need to consider *uniform limits of formulae*, which we call definable predicates:

Definition

Suppose that $\mathcal{M}$ is a structure and $A \subseteq M$. Then $P : M^n \to [0, 1]$ is said to be a *definable predicate in $\mathcal{M}$ over A* if there are formulae $\varphi_n(x)$ with parameters from A such that the sequence $(\varphi_n^{\mathcal{M}})$ converges uniformly to P.

Remark

Although each φ_n can only mention finitely many parameters from A, the sequence (φ_n) can mention *countably* many parameters from A. Thus, definable things (sets, functions,...) are always definable over countably many parameters, but not necessarily finitely many parameters.

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Definable functions

1. For $A \subseteq M$, a function $f : M^n \to M$ is *A-definable* if the predicate $(x, y) \mapsto d(f(x), y) : M^{n+1} \to [0, 1]$ is an A-definable predicate.
2. Given an elementary extension $\mathcal{M} \preceq \mathcal{N}$, such a function admits a canonical extension $\tilde{f} : N^n \to N$, which is also A-definable: We have $(\varphi_n^{\mathcal{M}})$ converging uniformly to $d(f(x), y)$. Then $(\varphi_n^{\mathcal{N}})$ will converge uniformly to some $Q(x, y)$. One then checks that the zeroset of Q defines a function, which will be $\tilde{f}$.

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2. Given an elementary extension $\mathcal{M} \preceq \mathcal{N}$, such a function admits a canonical extension $\tilde{f} : N^n \to N$, which is also A-definable.
3. Definable functions are uniformly continuous.
4. If $f : M^n \to M$ is A-definable, then for every $x = (x_1, \ldots, x_n) \in M^n$, we have $f(x) \in \mathrm{dcl}(A \cup \{x_1, \ldots, x_n\})$.

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Definable functions in $\mathfrak{U}$

Again, $\mathbb{U}$ is an ω_1-saturated elementary extension of $\mathfrak{U}$.

Theorem (G.)

If $f : \mathfrak{U}^n \to \mathfrak{U}$ is A-definable, then either $\tilde{f}$ is a projection function $(x_1, \ldots, x_n) \mapsto x_i$ or else $\tilde{f}$ has compact image contained in $\bar{A} \subseteq \mathfrak{U}$. Consequently, either f is a projection function or else has relatively compact image.

Corollaries

Corollary

1. *If* $f : \mathfrak{U} \to \mathfrak{U}$ *is a definable surjective/open/proper map, then* $f = \mathrm{id}_{\mathfrak{U}}$.
2. *If* $f : \mathfrak{U} \to \mathfrak{U}$ *is a definable isometric embedding, then* $f = \mathrm{id}_{\mathfrak{U}}$.
3. *If* $n \geq 2$, *then there are no definable isometric embeddings* $\mathfrak{U}^n \to \mathfrak{U}$.

Reason: Compact sets in $\mathfrak{U}$ have no interior.

Isometric Embeddings $\mathfrak{U} \to \mathfrak{U}$

There are many natural isometric embeddings $\mathfrak{U} \to \mathfrak{U}$, none of which (other than $\mathrm{id}_{\mathfrak{U}}$) are definable in $\mathfrak{U}$.

Examples

1. Suppose that X_1 and X_2 are compact subspaces of $\mathfrak{U}$. Then any isometry $\phi : X_1 \to X_2$ can be extended to an isometry $\tilde{\phi} : \mathfrak{U} \to \mathfrak{U}$.
2. Suppose that $x_1, \ldots, x_n \in \mathfrak{U}$. Define
$$\mathrm{Med}(x_1, \ldots, x_n) := \{z \in \mathfrak{U} \mid d(z, x_i) = d(z, x_j) \text{ for all } i, j\}.$$
Then $\mathrm{Med}(x_1, \ldots, x_n)$ is isometric to $\mathfrak{U}$.
3. Suppose that M is a Polish subspace of $\mathfrak{U}$ which is a Heine-Borel subspace. Then for any $R \in (0, 1]$, $\{x \in \mathfrak{U} \mid d(x, M) \geq R\}$ is isometric to $\mathfrak{U}$.

Definable Groups

Corollary

There are no definable group operations on $\mathfrak{U}$.

Cameron and Vershik introduced a group operation on $\mathfrak{U}$ for which there is a dense cyclic subgroup. This group operation allows one to introduce a notion of translation in $\mathfrak{U}$. By the above corollary, this group operation is not definable.

Key Ideas to the Proof for $n = 1$

Suppose that $f : \mathfrak{U} \to \mathfrak{U}$ is an A-definable function, where $A \subseteq \mathfrak{U}$ is countable. Let $\tilde{f} : \mathbb{U} \to \mathbb{U}$ denote its canonical extension.

1. By triviality of dcl, for any $x \in \mathbb{U}$, we have $\tilde{f}(x) \in \mathrm{dcl}(Ax) = \bar{A} \cup \{x\}$.
2. Let $X = \{x \in \mathfrak{U} \mid f(x) = x\}$. Show that $\tilde{f}^{-1}(\bar{A}) \setminus X \subseteq \mathrm{int}(\tilde{f}^{-1}(\bar{A}))$.
3. Prove a general lemma showing that if $F \subseteq \mathbb{U}$ is a closed subset and $G \subseteq F$ is a closed, *separable* subset of F for which $F \setminus G \subseteq \mathrm{int}(F)$, then either $F = G$ or $F = \mathbb{U}$. This involves a bit of "Urysohn-esque" arguing.
4. Finally, a saturation argument shows that if $\tilde{f}(\mathbb{U}) \subseteq \mathfrak{U}$, then $\tilde{f}(\mathbb{U})$ is compact.

Proof of Step 2

Lemma

$X = \{x \in \mathfrak{U} \mid f(x) = x\}$. *Then* $\tilde{f}^{-1}(\bar{A}) \setminus X \subseteq \mathrm{int}(\tilde{f}^{-1}(\bar{A}))$.

Proof.

Suppose $\tilde{f}(x) \in \bar{A}$ and $\tilde{f}(x) \neq x$. Let $r := d(\tilde{f}(x), x) > 0$. Let $\delta = \min(\frac{r}{2}, \Delta_f(\frac{r}{2}))$. Suppose $d(x, y) < \delta$. Then $d(\tilde{f}(x), \tilde{f}(y)) \leq \frac{r}{2}$. If $\tilde{f}(y) = y$, then

$$d(x, \tilde{f}(x)) \leq d(x, y) + d(\tilde{f}(x), y) < r,$$

a contradiction. Thus $y \in \tilde{f}^{-1}(\bar{A})$. □

Urysohn-esque arguing

Lemma

Let $(x_i \mid i < \omega)$ be a sequence from $\mathbb{U}$ and $(r_i \mid i < \omega)$ a sequence from $(0,1)$. Set $B := \bigcup_{i<\omega} B(x_i; r_i)$. Then $\mathbb{U} \setminus B$ is finitely injective.

Proof

Fix $a_1, \ldots, a_n \in \mathbb{U} \setminus B$ and let $\{a_1, \ldots, a_n, y\}$ be a one-point metric extension. By saturation, it is enough to find, for each $m < \omega$, a $z \in \mathbb{U}$ such that $d(y, a_i) = d(z, a_i)$ for $i = 1, \ldots, n$ and such that $d(z, x_i) \geq r_i$ for each $i = 1, \ldots, m$.

Urysohn-esque arguing (cont'd)

Proof (cont'd)

Consider the one-point metric exension

$$\{a_1, \ldots, a_n, x_1, \ldots, x_m, z\}$$

of $\{a_1, \ldots, a_n, x_1, \ldots, x_m\}$ given by:

- $d(z, a_i) = d(y, a_i)$ for each $i \in \{1, \ldots, n\}$, and
- $d(z, x_j) = \min_{1 \leq k \leq n}(d(y, a_k) + d(a_k, x_j))$ for each $j \in \{1, \ldots, m\}$.

Such a z can be found in $\mathbb{U}$ and this z is as desired. □

Corollary

$\mathbb{U} \setminus B$ is path-connected.

Proof of Step 3

Lemma

Suppose that $F \subseteq \mathbb{U}$ is closed and $G \subseteq F$ is a closed, separable subset of F for which $F \setminus G \subseteq \mathrm{int}(F)$. Then either $F = G$ or $F = \mathbb{U}$.

Proof.

Suppose $F \neq G$. Let $0 < r < d(y, G)$. Cover G with countably many balls of radius r and call the union of these balls B. Set $Y = \mathbb{U} \setminus B$, which is path-connected by the previous lemma. Now $F \cap Y = \mathrm{int}(F) \cap Y$ is a nonempty, clopen subset of Y, implying that $F \cap Y = Y$. It follows that $Y \subseteq F$. Since r can be taken to be arbitrarily small, this shows that $\mathbb{U} \setminus G \subseteq F$, whence $F = \mathbb{U}$. $\square$

Proof of Step 4

Lemma

Suppose that $\tilde{f}(\mathbb{U}) \subseteq \mathfrak{U}$. Then $\tilde{f}(\mathbb{U})$ is compact.

Proof.

It is a fact that $\tilde{f}(\mathbb{U})$ is closed, so we only need to show that it is totally bounded. Fix $\delta > 0$. Let $\varphi(x, y)$ be a formula that approximates $d(f(x), y)$ with error $\frac{\delta}{4}$. Let $(a_i \ : \ i < \omega)$ be a dense subset of $\mathfrak{U}$. Then the collection $\{\varphi(x, a_i) \geq \frac{\delta}{2} \ : \ i < \omega\}$ of conditions is *inconsistent*. By ω_1-saturation, there are $a_1, \ldots, a_n$ such that $\{\varphi(x, a_i) \geq \frac{\delta}{2} \ : \ 1 \leq i \leq n\}$ is inconsistent. It follows that $\tilde{f}(\mathbb{U}) \subseteq \bigcup_{i=1}^n B(a_i; \delta)$. $\square$

Question

Question 3

Can we improve the theorem on definable functions to read: If $f : \mathfrak{U}^n \to \mathfrak{U}$ is definable, then either f is a projection or a constant function?

I can show that a positive solution to the above question follows from a positive solution to the $n = 1$ case.

The Case of Relatively Compact Image

In the hopes of answering this question, we can say some things about $\tilde{f}(\mathbb{U}^n)$ in the case that it is relatively compact:

- $\tilde{f}(\mathbb{U}^n)$ is a continuum (connected, compact space).
- Consequently, if $\bar{A}$ is totally disconnected, then $\tilde{f}$ is a constant function.
- $\tilde{f}(\mathbb{U}^n)$ is a perfect space unless it is a singleton.
- If $\tilde{f}(\mathbb{U}^n)$ is not a singleton, then $\tilde{f}(\mathbb{U}^n)$ is either a Peano space (continuous image of $[0, 1]$) or else a reducible continuum (every two points are contained in a proper subcontinuum.)
- Consequently, $\tilde{f}(\mathbb{U}^n)$ is a decomposable continuum. Since the generic continuum is (hereditarily) indecomposable, we see that $\tilde{f}(\mathbb{U}^n)$ is a special kind of continuum.
- $\tilde{f}(\mathbb{U}^n)$ contains *arbitrarily small path-connected subcontinua*.

Question

Question 4

Are there any definable injections $f : \mathfrak{U} \to \mathfrak{U}$ other than the identity?

There can exist injective functions $\mathfrak{U} \to \mathfrak{U}$ which have relatively compact image, so our theorem doesn't immediately help us: Consider

$$(x_n) \mapsto (\frac{x_n}{2^n}) : (0,1)^\infty \to \ell^2.$$

and use the fact that $\mathfrak{U} \cong \ell^2 \cong (0,1)^\infty$.

Observe that a positive answer to Question 3 yields a negative answer to this question.

Injective Definable Functions

Lemma

If $f : \mathbb{U} \to \mathbb{U}$ is injective and definable, then $f = \mathrm{id}_{\mathbb{U}}$.

Proof.

One can show that the complement of an open ball in $\mathbb{U}$ is definable. Since f maps definable sets to definable sets (which is a fact we are unsure of in $\mathfrak{U}$), it follows that f is a closed map, whence a topological embedding. By our main theorem, we see that f is the identity. □

Remark

This doesn't immediately help us, for an injective definable map $\mathfrak{U} \to \mathfrak{U}$ need not induce an injective definable map $\mathbb{U} \to \mathbb{U}$. (Continuous logic is a positive logic!)

Upwards Transfer

Lemma (BBHU, Ealy-G.)

Suppose that M is ω-satuated and $P, Q : M^n \to [0,1]$ are definable predicates such that P is defined over a finite parameterset. Then the statement " for all $a \in M^n$ $(P(a) = 0 \Rightarrow Q(a) = 0)$" is expressible in continuous logic.

- It follows that the natural extension of an isometric embedding is also an isometric embedding:

$$|d(x,y) - r| = 0 \Rightarrow |d(f(x), f(y)) - r| = 0.$$

- It also follows that if $f : M^n \to M$ is an A-definable injection, where A is *finite*, then $\tilde{f}$ is also an injection:

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Musings on Definable Sets

A closed set $X \subseteq \mathfrak{U}^m$ is *A-definable* if the predicate $x \mapsto d(x, X) : \mathfrak{U}^m \to [0, 1]$ is A-definable.

By the strong ω-categoricity of $T_{\mathfrak{U}}$, we have that, for finite $A \subseteq \mathfrak{U}$, $X \subseteq \mathfrak{U}^m$ is A-definable if and only if X is invariant under $\mathrm{Aut}(\mathfrak{U}/A)$. Consequently, for A-definable $X, Y \subseteq \mathfrak{U}$, we have:

- ∂X, $\overline{\mathrm{int}(X)}$, $\overline{\mathfrak{U} \setminus X}$, $X \cap Y$, and $\mathrm{Ker}(X)$ are A-definable.
- If X is connected, then X is a "generalized annulus".
- The connected components of X are A-definable and any 1-element connected subset of X must be an element of A. Moreover, if there are infinitely many connected components of X, then they cannot be a uniform distance apart.
- If X is compact, then X is a (finite) subset of A.

Question

Question 5

What can we say about arbitrary definable subsets of $\mathfrak{U}^m$?

There probably is no nice "geometric" description of the definable sets. Indeed, any compact set is definable in any metric structure, so any compact metric space is a definable subset of $\mathfrak{U}$. However, maybe we can obtain results along the lines of the preceding slide saying that certain topological and geometric constructions preserve definability...

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