Nonstandard methods in Lie theory

Isaac Goldbring

UCLA

University of Illinois at Chicago Math Department Colloquium
January 20, 2012

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Nonstandard Extensions

Start with a mathematical universe V containing all relevant mathematical objects, e.g.

- $\mathbb{N}$, $\mathbb{R}$, a topological group G, a Lie algebra $\mathfrak{g}$;
- various cartesian products of the above sets;
- the elements of the above sets and the power sets of the above sets;

Then extend, functorially, to a *nonstandard* mathematical universe V^*:

- To every $A \in V$, there is a corresponding $A^* \in V^*$, e.g. we have $\mathbb{N}^*$, $\mathbb{R}^*$, G^*, $\sin^*(x)$ (which is a function $\mathbb{R}^* \to [-1, 1]^*$);
- For simplicity, we write sin for $\sin^*$.
- (Transfer Principle) V^* should behave "logically" like V.

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- For simplicity, we write sin for $\sin^*$.
- (Transfer Principle) V^* should behave "logically" like V.

The Transfer Principle

Example

Let $(G, \cdot, 1)$ be a group. Then the following are true in V:

- $(\forall x \in G)(\exists y \in G)[(x \cdot y = 1)$ and $(y \cdot x = 1)]$
- $(\forall x \in G)(x \cdot 1 = 1 \cdot x = x)$
- $(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$.

By transfer, the following are true in V^*:

- $(\forall x \in G^*)(\exists y \in G^*)[(x \cdot y = 1)$ and $(y \cdot x = 1)]$
- $(\forall x \in G^*)(x \cdot 1 = 1 \cdot x = x)$
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Ultrapowers

Definition

Given an infinite set I, a *nonprincipal ultrafilter on I* is a finitely additive $\{0, 1\}$-valued probability measure μ on I such that finite sets get measure 0.

Given a nonprincipal ultrafilter μ on I and a family $(X_i)_{i \in I}$ of sets, we can form their *ultraproduct* $\prod_\mu X_i$, which is the quotient of $\prod_i X_i$ by the relation of μ-a.e. agreement. If each $X_i = X$, then we refer to $X^\mu := \prod_\mu X$ as an *ultrapower of X*.

Fact

For any infinite set I and nonprinicpal ultrafilter μ on I, X^μ is a nonstandard extension of X.

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$\mathbb{R}^\mu$

Consider the example of $\mathbb{R}^\mu$. The transfer principle says that $\mathbb{R}^\mu$ is an ordered field extension of $\mathbb{R}$. For example:

- Suppose that $x = [(x_i)]_\mu \neq 0$. Then $x_i \neq 0$ a.e. Define $y_i := x_i^{-1}$ when $x_i \neq 0$ and otherwise define $y_i :=$ who cares . Then $x_i \cdot y_i = 1$ a.e., so $y := [(y_i)]_\mu = x^{-1}$.
- Similarly, define $x < y$ if and only if $x_i < y_i$ a.e. This is a linear order because the whole set has measure 1.

Suppose $I = \mathbb{N}$. Then $\alpha := [(1, \frac{1}{2}, \frac{1}{3}, \ldots)] \in \mathbb{R}^\mu$ is a *positive infinitesimal* (because finite sets have measure 0) and $\frac{1}{\alpha}$ is a positive *infinite element*. In fact, $\frac{1}{\alpha} = [(1, 2, 3, \ldots,)] \in \mathbb{N}^*$.

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Internal sets and saturation

$A \subseteq X^\mu$ is *internal* if there are $A_i \subseteq X$ such that $A = \prod_\mu A_i$.
More generally, $A \subseteq X^*$ is internal if $A \in \mathcal{P}(X)^*$.

Definition

Let κ be an infinite cardinal. We say that V^* is κ-**saturated** if whenever $\{\mathcal{O}_i \mid i < \kappa\}$ is a family of *internal* sets such that any intersection of a finite number of them is nonempty, then the intersection of all them is nonempty.

We will assume our V^* is κ-saturated for a suitably large κ. This can be arranged by choosing suitable I and μ.

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An Example of Saturation: Infinitesimals Again

For $i \in \mathbb{N}$, let $\mathcal{O}_i := \{x \in \mathbb{R}^* \mid 0 < x < \frac{1}{i}\}$. Each $\mathcal{O}_i$ is internal and any finite intersection of the $\mathcal{O}_i$ is nonempty. Saturation yields $x \in \bigcap_{i \in \mathbb{N}} \mathcal{O}_i$. Such an x is a positive infinitesimal.

The set of infinitesimals is *external*, for otherwise it would have a least upper bound (by Transfer), which it doesn't. Similarly, the set of infinite elements is external.

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Infinitesimals in Hausdorff Spaces

- Suppose X is a Hausdorff topological space. Then one can use a similar trick as in the previous slide to construct for any $a \in X$ an element a' such that $a' \in U^*$ for all neighborhoods U of a in X. Such an a' is infinitely close to a and we write $a' \in \mu(a)$. (Usually external)
- We let $X_{\mathrm{ns}} := \bigcup_{a \in X} \mu(a)$, the set of *nearstandard* elements of X^*. (Usually external)
- The Hausdorff axiom implies that $\mu(a) \cap \mu(b) = \emptyset$ if $a \neq b$. Thus, if $a' \in X_{\mathrm{ns}}$, we can write $\mathrm{st}(a')$, the *standard part* of a', for the unique element of X that a' is infinitely close to. If $a, b \in X_{\mathrm{ns}}$ are such that $\mathrm{st}(a) = \mathrm{st}(b)$, we sometimes write $a \sim b$.
- In the case of a topological group G, we will denote $\mu(1)$ simply by μ and will call it the set of *infinitesimals* of G.

Infinitesimals in Hausdorff Spaces

Lemma

If X is a Hausdorff space, $U \subseteq X$, and $a \in U$, then a is in the interior of U if and only if $\mu(a) \subseteq U^*$.

Lemma

Suppose X and Y are Hausdorff spaces, $f : X \to Y$ is a function, and $a \in X$. Then f is continuous at a if and only if $f(\mu(a)) \subseteq \mu(f(a))$.

Lemma (Robinson)

X is compact if and only if $X^* = X_{\text{ns}}$.

Hilbert's Fifth Problem

Definition

A topological group G is **locally euclidean** if there is a neighborhood of the identity homeomorphic to some $\mathbb{R}^n$.

Definition

G is a **Lie group** if G is a real analytic manifold which is also a group such that the maps $(x, y) \mapsto xy : G \times G \to G$ and $x \mapsto x^{-1} : G \to G$ are real analytic maps.

Hilbert's Fifth Problem (H5)

If G is a locally euclidean topological group, is there a real analytic structure on G compatible with the topology so that the group operations become real analytic?

Positive Answers to H5

- Linear Case: G can be continuously embedded into $\mathrm{Gl}_n(\mathbb{R})$ for some n (von Neumann)
- Abelian Case (Pontrjagin)
- Compact Case (Weyl)
- Full Solution: Gleason, Montgomery, Zippin (1952)

Theorem

For a locally compact (Hausdorff) group G, the following are equivalent:

1. *G is locally euclidean;*
2. *G has* ***no small subgroups****, i.e. there is a neighborhood of the identity containing no nontrivial subgroups of G;*
3. *G is a Lie group.*

- Nonstandard Exposition of the Full Solution: Hirschfeld (1990)

One Parameter Subgroups

Definition

A **one-parameter subgroup of** G (1-ps of G) is a continuous group morphism $X : \mathbb{R} \to G$.

Put $L(G) := \{X : \mathbb{R} \to G \mid X \text{ is a 1-ps of } G\}$.

We have the **scalar multiplication** map
$(r, X) \mapsto rX : \mathbb{R} \times L(G) \to L(G)$, where $(rX)(t) := X(rt)$.

We let O denote the trivial 1-ps of G, i.e. $O(t) \equiv 1$. Then:

- $0X = O$ and $1X = X$;
- $r(sX) = (rs)X$.

The Case of Lie Groups

- Suppose G is a Lie group and $X \in L(G)$. Then X is real analytic and so $X'(0) \in T_1(G)$. We get a bijection $X \mapsto X'(0) : L(G) \to T_1(G)$ and the addition operation on $L(G)$ that makes this an $\mathbb{R}$-vector space isomorphism is given by

$$(X + Y)(t) = \lim_{n\to\infty} (X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}.$$

- Let $n = \dim G = \dim_{\mathbb{R}} L(G)$ and make $L(G)$ a real analytic manifold so that the linear isomorphisms $L(G) \cong \mathbb{R}^n$ are analytic isomorphisms.
- Then the **exponential map** $X \mapsto X(1) : L(G) \to G$ yields an analytic isomorphism from an open neighborhood of O in $L(G)$ onto an open neighborhood of 1 in G.

Plan of Proof for NSS implies Lie

Suppose G is locally compact and has NSS. One takes the following steps to prove that G is a Lie group.

1. Show that for any $X, Y \in L(G)$, there is an $X + Y \in L(G)$ given by

$$(X+Y)(t) = \lim_{n\to\infty}(X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}$$

   and that $L(G)$ becomes an $\mathbb{R}$-vector space under this addition and the aforementioned scalar multiplication. (In this talk, we will just sketch the proof of the existence of $X + Y$.)

2. Equip $L(G)$ with its compact-open topology and show that $L(G)$ becomes a topological vector space.

Plan of Proof for NSS implies Lie (cont'd)

3. Show that the exponential map

$$X \mapsto X(1) : L(G) \to G$$

   maps an open neighborhood of O in $L(G)$ onto an open neighborhood of 1 in G. Then since G is locally compact, so is $L(G)$, whence we conclude that $\dim_{\mathbb{R}}(L(G)) < \infty$. (This also shows that G is locally euclidean.)
4. Use the adjoint representation of G on $L(G)$ to make G into a Lie group.

Adjoint Representation

- Let $\mathrm{Ad} : G \to \mathrm{Aut}(L(G))$ be defined by $\mathrm{Ad}(g)(X) = aXa^{-1}$, where $(aXa^{-1})(t) := aX(t)a^{-1}$. Then Ad is a continuous group morphism.
- Since $G/\ker(\mathrm{Ad})$ continuously embeds into $\mathrm{Aut}(L(G)) \cong \mathrm{Gl}_n(\mathbb{R})$, where $n := \dim_{\mathbb{R}}(L(G))$, von Neumann's Theorem implies that $G/\ker(\mathrm{Ad})$ is a Lie group.
- If G is connected (which we may suppose it is), then $\ker(\mathrm{Ad}) = \mathrm{center}(G)$, which is abelian (and NSS), and hence a Lie group by Pontrjagin.
- Now use a result of Kuranishi: if $1 \to N \to G \to G/N \to 1$ is an exact sequence that admits "local cross sections", where N is an abelian Lie group and G/N is a Lie group, then G is a Lie group.

Powers of Infinitesimals

If $a, b \in \mu$, then $a \cdot b \sim e \cdot e = e$, i.e. $ab \in \mu$. It then follows that $a^n \in \mu$ for any $n \in \mathbb{N}$. What about infinite powers of a?

Internal Induction

If $A \subseteq \mathbb{N}^*$ is *internal*, contains 0 and is closed under the successor operation, then $A = \mathbb{N}^*$.

Let $a \in \mu$ and let $A = \{\sigma \in \mathbb{N}^* \mid a^\sigma \in \mu\}$. Then A contains 0 and is closed under successor by continuity of multiplication. The problem is that A is not internal since μ is an *external* set. Hence we cannot conclude that $a^\sigma \in \mu$ for all $\sigma \in \mathbb{N}^*$. (In fact, G is NSS if and only if whenever $a \in \mu$ satisfies $a^\sigma \in \mu$ for all $\sigma \in \mathbb{N}^*$, it was because $a = 1$.)

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Landau Notation

We will need to use the following notation:

Notation

Suppose $i \in \mathbb{Z}^*$ and $\sigma \in \mathbb{N}^* \setminus \{0\}$.

- Say $i = \mathrm{o}(\sigma)$ if $|i| < \frac{\sigma}{n}$ for every $n \in \mathbb{N}$;
- Say $i = \mathrm{O}(\sigma)$ if $|i| < n\sigma$ for some $n \in \mathbb{N}$.

Infinitesimal Generators of 1-ps's

Fix $\sigma \in \mathbb{N}^* \setminus \mathbb{N}$.

Definition

Let $G(\sigma) := \{a \in \mu \mid a^i \in \mu \text{ for all } i = o(\sigma)\}$. Note that $1 \in G(\sigma)$ and $G(\sigma)$ is closed under inverses.

Fact

If $a \in G(\sigma)$, then $a^i \in G_{ns}$ for all $i = O(\sigma)$.

Definition

For $a \in G(\sigma)$, define $X_a : \mathbb{R} \to G$ by $X_a(t) := \mathrm{st}(a^{[t\sigma]})$.

Fact

X_a is a 1-ps of G.

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Infinitesimal Generators of 1-parameter subgroups of G (cont'd)

Question

Which elements of $L(G)$ are of the form X_a for some $a \in G(\sigma)$?

Answer

All of them! Suppose $X \in L(G)$ and let $a := X(\frac{1}{\sigma}) \in \mu$. Then if $i = o(\sigma)$, we have $a^i = X(\frac{i}{\sigma}) \in \mu$, whence $a \in G(\sigma)$, and

$$X_a(t) = \mathrm{st}(a^{[t\sigma]}) = \mathrm{st}(X(\frac{[t\sigma]}{\sigma})) = X(t).$$

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Infinitesimal Generators of 1-parameter subgroups of G (cont'd)

One can ask the question "When does $X_a = O$?" This happens if and only if $a^i \in \mu$ for all $i = \mathrm{O}(\sigma)$.

Definition

$G^o(\sigma) := \{a \in \mu \mid a^i \in \mu \text{ for all } i = \mathrm{O}(\sigma)\}$.

So we have $X_a = O$ if and only if $a \in G^o(\sigma)$ and observe that $1 \in G^o(\sigma)$, $G^o(\sigma) \subseteq G(\sigma)$ and $G^o(\sigma)$ is closed under inverses.

Gleason-Yamabe Lemmas

One next wants to show that $G(\sigma)$ is a subgroup of μ with normal subgroup $G^o(\sigma)$ and use this to help us put a group law on $L(G)$. To do this and more, we will need to know more about the growth rates of powers of elements of these sets. The ingenious idea of Hirschfeld was to translate the very technical lemmas of Gleason and Yamabe into clear and concise statements about such growth rates.

Lemma

- *Let $a_1, \ldots, a_\sigma$ be an internal sequence in G^* such that all $a_i \in G^o(\sigma)$. Then $a_1 \cdots a_\sigma \in \mu$.*
- *If $a \in G(\sigma)$ and $b \in G^o(\sigma)$, then $(ab)^i \sim a^i$ for all $i \leq \sigma$.*
- *Suppose $a, b \in G(\sigma)$ are such that $a^i \sim b^i$ for all $i \leq \nu$. Then $a^{-1}b \in G^o(\sigma)$.*

Consequences of the Gleason-Yamabe Lemmas

The following theorem follows rather easily from the aforementioned consequences of the Gleason-Yamabe Lemmas.

Theorem

- *$G(\sigma)$ is a group and $G^o(\sigma)$ is a normal subgroup;*
- *The quotient group $G(\sigma)/G^o(\sigma)$ is abelian;*
- *For $a, b \in G(\sigma)$, $X_a = X_b$ if and only if $a^{-1}b \in G^o(\sigma)$;*
- *The surjective map $a \mapsto X_a : G(\sigma) \to G^o(\sigma)$ induces a bijection $aG^o(\sigma) \mapsto X_a : G(\sigma)/G^o(\sigma) \to L(G)$.*

Group Law on $L(G)$

We make $L(G)$ into an abelian group with the operation $+_\sigma$ so that the aforementioned bijection is an abelian group isomorphism:

$$X_a +_\sigma X_b := X_{ab}, \text{ i.e. } (X +_\sigma Y)(t) = \mathrm{st}((X(\frac{1}{\sigma})Y(\frac{1}{\sigma}))^{[\sigma t]}).$$

Fact

If $\sigma, \nu \in \mathbb{N}^* \setminus \mathbb{N}$, then $X +_\sigma Y = X +_\nu Y$ for all $X, Y \in L(G)$.

Corollary

$X + Y$ exists in $L(G)$, i.e. $\lim_{n\to\infty}(X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}$ *exists for every* $t \in \mathbb{R}$.

This follows from the aforementioned fact, using that a sequence (a_n) from G converges if and only if $a_\sigma \sim a_\nu$ for all $\sigma, \nu \in \mathbb{N}^* \setminus \mathbb{N}$.

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Local groups

Definition

A **local group** is a tuple $(G, 1, \iota, p)$ where:

- G is a Hausdorff topological space, $1 \in G$;
- $\iota : \Lambda \to G$ is continuous, where $\Lambda \subseteq G$ is open;
- $p : \Omega \to G$ is continuous, where $\Omega \subseteq G \times G$ is open;
- $1 \in \Lambda$, $\{1\} \times G \subseteq \Omega$, $G \times \{1\} \subseteq \Omega$;
- $p(1, x) = p(x, 1) = x$;
- if $x \in \Lambda$, then $(x, \iota(x)) \in \Omega$, $(\iota(x), x) \in \Omega$, and

$$p(x, \iota(x)) = p(\iota(x), x) = 1;$$

- if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then

$$p(p(x, y), z) = p(x, p(y, z)).$$

A Simple Example of a Local Group

Let $G = (-1, 1)$. Then G is a local group under addition, where we take $\Lambda = G$ and $\Omega = \{(x, y) \in G \mid x + y \in G\}$.

More generally, if H is a topological group and U is an open neighborhood of the identity, then we obtain a local group $G := U$, where the local group operations are those inherited from H, $\Lambda = \{x \in G \mid x^{-1} \in G\}$ and $\Omega = \{(x, y) \in G \times G \mid xy \in G\}$.

These are very special types of local groups, called *globalizable* local groups.

Not every local group is globalizable

Consider the local Lie group given by $G = \mathbb{R}$,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid |xy| \neq 1\}, \quad \Lambda = G \setminus \{\frac{1}{2}, 1\},$$

with multiplication and inversion

$$p(x, y) = \frac{2xy - x - y}{xy - 1}, \iota(x) = \frac{x}{2x - 1}.$$

(Note that 0 is the neutral element.) Then G is not globalizable since $p(x, 1) = p(1, x) = 1$ for all $x \neq \pm 1$.

Let $U = \{|x| < \frac{1}{2}\}$. Then $x \mapsto \frac{x}{x-1} : U \to \mathbb{R}$ shows that $G|U$ is globalizable.

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The Local H5

Theorem (G.)

If $(G, 1, \iota, p)$ *is a locally euclidean local group, then* G *is locally isomorphic to a Lie group.*

- Since not every local group is globalizable, the Local H5 is not an immediate Corollary of the H5. In fact, the statement "every locally euclidean local group is locally isomorphic to a group" is equivalent to the above theorem.
- In 1957, Jacoby claimed a proof of the Local H5, but his proof was discovered to be wrong by Plaut about 20 years ago. Jacoby essentially assumes that every local group G is **globally associative**: given any finite sequence of elements from G, if there are two ways of introducing parentheses such that both products thus formed exist, then the two products are in fact equal.

Associativity

Suppose a, b, c, d reside in a local group G and $a(b(cd))$ and $((ab)c)d)$ are both defined. Why aren't they necessarily equal, as Jacoby thought they were?

The usual calculation:

$$a(b(cd)) = (ab)(cd) = ((ab)c)d.$$

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G is globally associative if and only if G is globalizable.

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NSA and the Local H5

If one had to give a simple reason as to why NSA is especially useful in the local setting, it would be that $(\mu, p|\mu)$ is **an actual group.**

Moreover, an elementary saturation argument shows that for every $a \in \mu$, there is $N \in \mathbb{N}^* \setminus \mathbb{N}$ so that a^i is defined for all $i \leq N$ in the sense that all ways of forming parentheses around the constant sequence (a) of length N yield defined products that are all equal. In this way, one can mimic many of the arguments used in the proof of the H5, often with much more care needed and with proofs doubling in length.

Local Yamabe

Theorem (Yamabe)

If G is a locally compact group, then there is an open subgroup G' of G such that, for every open neighborhood U of the identity, there is a normal compact subgroup H of G contained in U such that G'/H is NSS (and hence a Lie group).

Theorem (G.)

If G is a locally compact local group, then there is a restriction G' of G and a compact normal subgroup N of G' such that G'/N has NSS (and hence is locally isomorphic to a Lie group).

Corollary (van den Dries, G.)

Every locally compact local group is locally isomorphic to a group.

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An Application: Approximate groups

Definition

Let G be a (discrete) group, A a finite subset of G, and $K \in \mathbb{R}^{\geq 1}$. We say that A is a K*-approximate group* if $1 \in A$, $A = A^{-1}$, and $A \cdot A$ can be covered by at most K translates of A.

Example

A 1-approximate group is the same as a finite subgroup.

Example

If $(G, +)$ is abelian, $v_1, \ldots, v_r \in G$, and $N_1, \ldots, N_r \in \mathbb{R}^{>0}$, then the set $\{a_1 v_1 + \cdots + a_r v_r \ : \ a_1, \ldots, a_r \in \mathbb{Z}, |a_i| \leq N_i\}$ is called a *generalized arithmetic progression*. It is a 2^r-approximate group. (r is called the *rank* of the progression.)

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Freiman's Theorem

Theorem (Freiman)

If $(G, +)$ is a torsion-free abelian group and A is a K-approximate group in G, then there is a generalized arithmetic progression P of rank $O_K(1)$ in G such that $P \subseteq 4A$, $|A| \ll_K |P|$.

Theorem (Green-Rusza)

If $(G, +)$ is an abelian group and A is a K-approximate group in G, then there is a finite subgroup H of G and a generalized arithmetic progression P of rank $O_K(1)$ in G/H such that $\pi^{-1}(P) \subseteq 4A$ and $|A|/|H| \ll_K |P|$. (Here, $\pi : G \to G/H$.)

What about approximate groups in nonabelian groups?

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What about approximate groups in nonabelian groups?

Noncommutative progressions

Let G be a group.

Definition

If $a_1, \ldots, a_r \in G$ and $N_1, \ldots, N_r \in \mathbb{R}^{>0}$, then the set of all words in the alphabet $a_1^{\pm 1}, \ldots, a_r^{\pm 1}$ such that the total number of occurrences of a_i and a_i^{-1} combined are no more than N_i is called a *noncommutative progression*.

Noncommutative progressions need not be approximate groups (think free groups!). However, if $a_1, \ldots, a_r$ generate a nilpotent group of step s, then the noncommutative progression (which is then called a *nilprogression*) is an $O_{r,s}(1)$-approximate group.

Classifying Approximate Groups

Theorem (Breuillard, Green, Tao; Hrushovski)

If A is a K-approximate group in G, then there is a finite subgroup H of G and a nilprogression P of rank $O_K(1)$ and step $O_K(1)$ in $N(H)/H$ such that $\pi^{-1}(P) \subseteq A^4$ and $|A|/|H| \ll_K |P|$. ($\pi : N(H) \to N(H)/H$.)

$\pi^{-1}(P)$ is called a *coset nilprogression*.

Actually, this theorem holds for approximate groups in (discrete) local groups, once this is suitably defined. This is not merely to make things more general; this passage to local groups is crucial in the proof to prevent "accidentally large torsion."

Key ideas in the proof

- Suppose that the theorem is false. Then one takes an ultraproduct of counterexamples to obtain a so-called *ultra approximate group*.
- To obtain a contradiction, one needs to show that an ultra approximate group $\mathbb{A}$ contains a nondegenerate ultra coset nilprogression $\mathbb{P}$ with $|\mathbb{A}| < |\mathbb{P}|$ (as nonstandard natural numbers).
- Hrushrovski's idea was that an ultra approximate group $\mathbb{A}$ naturally admits a locally compact "model" $\mathbb{A} \to L$, which captures the "coarse" or "macroscopic" behavior of the ultra approximate group.
- Using the local Yamabe theorem, one can replace this locally compact model with a Lie model.
- Now one can use arguments by induction on $\dim(L)$ together with a certain "escape norm" on the ultra approximate group.

Banach-Lie groups and algebras

Definition

1. If E is a Banach space, then a Hausdorff space M is a *smooth E-manifold* if every point in M has an open neighborhood homeomorphic to an open set in E and such that the transition maps are smooth (in the sense of Frèchet). A *smooth Banach manifold* is a smooth E-manifold for some Banach space E.
2. A *Banach-Lie group* is a smooth Banach manifold which is also a group in which the group operations are smooth.
3. A *Banach-Lie algebra* is a Banach space $\mathfrak{g}$ which is also a Lie algebra and such that the bracket operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is continuous.

Enlargeable Banach-Lie Algebras

- As in the finite-dimensional situation, if G is a Banach-Lie group, then the tangent space at 1, $T_1(G)$, can naturally be equipped with the structure of a Banach-Lie algebra.
- However, unlike the finite-dimensional situtation, it is not the case that every Banach-Lie algebra is the Lie algebra of a Banach-Lie group. (van-Est, Korthagen)

Definition

A Banach-Lie algebra is called *enlargeable* if it is isomorphic to the Banach-Lie algebra of a Banach-Lie group.

How can we tell if a Banach-Lie algebra is enlargeable?

Pestov's Theorem on Localizing Enlargeability

Theorem (Pestov)

Suppose that $\mathfrak{g}$ *is a Banach-Lie algebra. Suppose that there exists a directed family* $\mathcal{H}$ *of closed subalgebras of* $\mathfrak{g}$ *and a neighborhood* V *of* 0 *in* $\mathfrak{g}$ *such that:*

1. $\bigcup \mathcal{H}$ *is dense in* $\mathfrak{g}$*;*
2. *Each* $\mathfrak{h} \in \mathcal{H}$ *is enlargeable and if* H *is the corresponding simply connected Banach-Lie group, then the exponential map* $\exp_H : \mathfrak{h} \to H$ *is injective when restricted to* V*.*

Then $\mathfrak{g}$ *is enlargeable.*

Corollary

If $\mathfrak{g}$ *is a Banach-Lie algebra with a dense subalgebra that is locally finite-dimensional or locally solvable, then* $\mathfrak{g}$ *is enlargeable.*

Idea of the Proof

- For an internal subalgebra $\mathfrak{h}$ of $\mathfrak{g}^*$ (e.g. $\mathfrak{h} = \mathfrak{g}^*$), set
$$\mathfrak{h}_{\text{fin}} := \{x \in \mathfrak{h} \mid \|x\| \text{ is finite}\}$$
and
$$\mu_{\mathfrak{h}} := \{x \in \mathfrak{h} \mid \|x\| \text{ is infinitesimal}\}.$$
- Then $\mathfrak{h}_{\text{fin}}$ is a real Lie algebra and $\mu_{\mathfrak{h}}$ is a Lie ideal of $\mathfrak{h}_{\text{fin}}$.
- We call the quotient Lie algebra $\hat{\mathfrak{h}} := \mathfrak{h}_{\text{fin}}/\mu_{\mathfrak{h}}$ the *nonstandard hull* of $\mathfrak{h}$. Equipping $\hat{\mathfrak{h}}$ with the norm $\|x + \mu_{\mathfrak{h}}\| := \text{st}(\|x\|)$, one can show, using saturation, that $\hat{\mathfrak{h}}$ is a Banach-Lie algebra.
- Note that there is an injective morphism of Banach-Lie algebras
$$\iota : \mathfrak{g} \to \widehat{\mathfrak{g}^*}.$$

Idea of the Proof (cont'd)

- Set $X := \bigcup \mathcal{H}$. By density of X and saturation, there is a *hyperfinite* subset A of X such that, for every $g \in \mathfrak{g}$, $X \cap \mu(g) \neq \emptyset$. (Hyperfinite means $A \in \mathcal{P}_{\text{fin}}(X)^*$, or A is an ultraproduct of finite sets.)
- Since $\mathcal{H}$ is directed, there is, by transfer, $\mathfrak{h} \in \mathcal{H}^*$ such that $A \subseteq \mathfrak{h}$.
- It follows that the embedding $\iota : \mathfrak{g} \to \hat{\mathfrak{g}^*}$ takes values in $\hat{\mathfrak{h}}$. It suffices to show that $\hat{\mathfrak{h}}$ is enlargeable (as closed subalgebras of enlargeable Banach-Lie algebras are enlargeable.)

Idea of the Proof (cont'd)

- By transfer, $\mathfrak{h}$ is internally enlargeable and if H is the corresponding internal Lie group with exponential map $\exp_H : \mathfrak{h} \to H$, then $\exp_H \restriction V^*$ is injective.
- Set $H_{\text{fin}} := \langle \exp_H(\mathfrak{h}_{\text{fin}}) \rangle$ and $\mu_H := \exp_H(\mu_{\mathfrak{h}})$. Using some facts about the BCH series, one can show that H_{fin} is a group and μ_H is a normal subgroup of H_{fin}. Set $\hat{H} := H_{\text{fin}}/\mu_H$. Then $\hat{H}$ is the *nonstandard hull of H*.
- One can show that there is a map $\widehat{\exp} : \hat{\mathfrak{h}} \to \hat{H}$ defined by $\widehat{\exp}(x + \mu_{\mathfrak{h}}) := \exp_H(x)\mu_H$. Using the injectivity of $\exp_H \restriction V^*$, one can show that $\widehat{\exp}$ is injective on a neighborhood of the origin in $\hat{\mathfrak{h}}$.
- Again, using the BCH series and $\widehat{\exp}$, one can make $\hat{H}$ into a Banach-Lie group for which $\hat{\mathfrak{h}}$ is its Lie algebra, proving that $\hat{\mathfrak{h}}$ is enlargeable.

Locally exponential Lie groups and algebras

- Banach-Lie theory is too specialized to cover natural examples from infinite-dimensional geometry, e.g. $C^\infty(M, G)$, where M is a compact manifold and G is a Banach-Lie group.
- The correct generality is to allow *locally convex spaces* as the model spaces (topological vector spaces whose topology is given by a family of seminorms).
- *Locally exponential Lie groups* are the Lie groups that possess a smooth exponential map that are moreover local diffeomorphisms.
- *Locally exponential Lie algebras* are the Lie algebras which are natural candidates to be the Lie algebra of a locally exponential Lie group.
- Neeb asked if Pestov's theorem holds for locally exponential Lie algebras.

An Analog of Pestov's Theorem

Theorem (G.)

Suppose $\mathfrak{g}$ is a locally exponential Lie algebra, $\mathcal{H}$ is a family of closed subalgebras of $\mathfrak{g}$, V is a neighborhood of 0 in $\mathfrak{g}$ and p is a continuous seminorm on $\mathfrak{g}$ satisfying:

1. *$\bigcup \mathcal{H}$ is dense in $\mathfrak{g}$;*
2. *for each $\mathfrak{h} \in \mathcal{H}$, there is a locally exponential Lie group H such that $L(H) \cong \mathfrak{h}$;*
3. *for each $\mathfrak{h} \in \mathcal{H}$, if H is a connected locally exponential Lie group such that $L(H) \cong \mathfrak{h}$, then $\exp_H | V \cap \mathfrak{h} : V \cap \mathfrak{h} \to H$ is injective;*
4. *$(\exp_H(\{x \in \mathfrak{h} | \, p(x) < 1\}))^2 \subseteq W_{\mathfrak{h}}$, where $W_{\mathfrak{h}}$ is an open neighborhood of 1 contained in $\exp_H(V)$;*
5. *m_U is uniformly continuous on $\{p < 1\}^{\times 2}$*
6. *m_U is uniformly smooth at finite points.*

Then $\mathfrak{g}$ is enlargeable.

Locally uniform groups

- More recently, I have noticed that a nonstandard hull construction is possible for topological groups that are uniformly continuous near the identity
- For simplicity, suppose that G is a metrizable group with left-invariant metric d whose multiplication is uniformly continuous on $B_d(e, \epsilon)$.
- Set $U_f := \{x \in G^* \mid \operatorname{st}(d(x,e)) < \epsilon\}$ and $\hat{U} := U_f / \approx$.
- Then $\hat{U}$ is a local group, called the *nonstandard hull of G.*
- One can show that, if G is a Banach-Lie group, then $\hat{U}$ is locally isomorphic to Pestov's nonstandard hull of G.

Locally uniform groups

Enflo wanted to pursue the study of Hilbert's fifth problem in infinite dimensions. His philosophy was that every concept needed a *uniform version*. For example:

Definition

A topological group G is *uniformly NSS* if there is a neighborhood U of the identity such that, for every neighborhood V of the identity, there is $n_V \in \mathbb{N}$ such that $x \notin V \Rightarrow x^n \notin U$ for some $n \leq n_V$.

Theorem (G.)

If G is a metrizable locally uniform group, then G is uniformly NSS if and only if $\hat{U}$ is NSS (for an appropriately chosen U).

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Nonstandard methods in infinite-dimensional Lie theory

Infinite-dimensional Lie theory (in its current form) is a very young subject with many interesting open problems. For example:

Locally compact subgroup problem

Suppose that G is an infinite-dimensional Lie group with locally compact subgroup H. Is H a (necessarily finite-dimensional) Lie group?

It is my hope that nonstandard methods (such as the nonstandard hull construction) will help answer some of these questions.

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