The theory of tracial von Neumann algebras does not have a model companion

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1 von Neumann algebras

2 Model companions

3 Model complete theories of tracial vNas

4 Independence relations
Hilbert spaces

Definition

A Hilbert space $H$ is a complex inner product space such that the induced metric is complete.

Examples

- $\mathbb{C}^n$, where $\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i y_i$.
- $\ell^2 = \{ (x_n) \in \mathbb{C}^\mathbb{N} : \sum_n |x_n|^2 < \infty \}$ where $\langle (x_n), (y_n) \rangle := \sum_{n=1}^{\infty} x_n \overline{y_n}$.
- $L^2(X, \mu) := \{ f : X \to \mathbb{C} : f \text{ is measurable and } \int_X |f|^2 \, d\mu < \infty \}$, where $\langle f, g \rangle := \int_X f \overline{g} \, d\mu$ (for $(X, \mu)$ a finite measure space).
Bounded operators

Definition

If $X$, $Y$ are normed spaces (over $\mathbb{C}$), then a linear transformation $T : X \to Y$ is *bounded* if the image of the unit ball of $X$ under $T$ is bounded.

- If $T$ is bounded, then we set $\| T \| := \sup \{ \| Tx \| : \| x \| = 1 \}$, called the *operator norm of $T$*, and observe that $\| T \|$ is the least upper bound for the image of the unit ball of $X$ under $T$.
- The set of bounded linear operators $\mathcal{B}(X, Y)$ from $X$ to $Y$ forms a normed space with the above notion of $\| T \|$.
- If $X = Y$, we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X, X)$.
- $T$ is bounded if and only if $T$ is (uniformly) continuous.
Examples of bounded operators

Examples

- Every linear transformation between finite-dimensional normed spaces is bounded.
- Fix \((d_n) \in \mathbb{C}^n\) and consider \(T : \mathbb{C}^n \to \mathbb{C}^n\) given by \(T((x_n)) := (d_n x_n)\). Then \(T \in B(\ell^2)\) if and only if \((d_n)\) is bounded.
- If \(f \in L^\infty(X, \mu)\), then \(m_f : L^2(X, \mu) \to L^2(X, \mu)\) defined by \(m_f(g) := fg\) is a bounded linear transformation.
Suppose that $H$ is a Hilbert space. We consider the following topologies on $B(H)$:

- The **operator norm topology**.
- The **strong topology**: a subbasis of open sets is given by
  \[ \{ T \in B(H) : \| T(v) - T_0(v) \| < \epsilon \}, \]
  where $T_0 \in B(H)$, $v \in H$ and $\epsilon > 0$.
- The **weak topology**: a subbasis of open sets is given by
  \[ \{ T \in B(H) : | \langle T(v) - T_0(v), w \rangle | < \epsilon \}, \]
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Notice norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.
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Notice norm convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence.
Why other topologies?

**Lemma**

1. The map $T \rightarrow T^*$ is weakly continuous but not strongly continuous.
2. The map $(S, T) \rightarrow ST$ is separately strongly continuous but not jointly strongly continuous.
3. If $A \subseteq \mathcal{B}(H)$ is a $*$-subalgebra, then so is the weak closure of $A$. 
von Neumann's bicommutant theorem

Given a subset $S$ of $\mathcal{B}(H)$, we let

$$S' := \{ T \in \mathcal{B}(H) :TU = UT \text{ for all } U \in S \}.$$ 

Notice that $S'$ is always a subalgebra of $\mathcal{B}(H)$ and $S \subseteq S''$ is always true.

**Theorem (von Neumann)**

Suppose that $A \subseteq \mathcal{B}(H)$ is a unital $\ast$-subalgebra. The following are equivalent:

- $A = S'$ for some $S \subseteq \mathcal{B}(H)$;
- $A = A''$;
- $A$ is closed with respect to the weak topology;
- $A$ is closed with respect to the strong topology.

A unital $\ast$-subalgebra of $\mathcal{B}(H)$ satisfying any of the equivalent conditions of the above theorem is called a **von Neumann algebra**.
Examples of vNas

Example

$\mathcal{B}(H)$ is a von Neumann algebra.

Example

Suppose that $(X, \mu)$ is a finite measure space. Then $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ by left multiplication, yielding an embedding

$$L^\infty(X, \mu) \hookrightarrow \mathcal{B}(L^2(X, \mu)),$$

the image of which is a von Neumann algebra. (Actually, all abelian von Neumann algebras are isomorphic to some $L^\infty(X, \mu)$, whence von Neumann algebra theory is sometimes dubbed “noncommutative measure theory.”)
Group von Neumann algebras

Example

Suppose that $G$ is a locally compact group and $\alpha : G \rightarrow B(H)$ is a unitary group representation. Then the group von Neumann algebra of $\alpha$ is $\alpha(G)^\prime \prime$. (Understanding $\alpha(G)^\prime \prime$ is tantamount to understanding the invariant subspaces of $\alpha$.)

In the important special case that $\alpha : G \rightarrow B(L^2(G))$ (where $G$ is equipped with its haar measure) is given by left translations

$$\alpha(g)(f)(x) := f(g^{-1}x),$$

we call $\alpha(G)^\prime \prime$ the group von Neumann algebra of $G$ and denote it by $L(G)$. 
Example

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Example

Let $M_2$ denote the set of $2 \times 2$ matrices with entries from $\mathbb{C}$. We consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \cdots$$

and set $M := \bigcup_{n=1}^{\infty} \bigotimes_n M_2$.

- The normalized traces on $\bigotimes_n M_2$ form a cohesive family of traces, yielding a trace $\text{tr} : M \to \mathbb{C}$.

- We can define an inner product on $M$ by $\langle A, B \rangle := \text{tr}(B^* A)$. Set $H$ to be the completion of $M$ with respect to this inner product.

- $M$ acts on $H$ by left multiplication, whence we can view $M$ as a $\ast$-subalgebra of $B(H)$. We set $\mathcal{R}$ to be the von Neumann algebra generated by $M$. $\mathcal{R}$ is called the hyperfinite $II_1$ factor.
Tracial von Neumann algebras

Suppose that $A$ is a von Neumann algebra. A *tracial state* (or just *trace*) on $A$ is a linear functional $\tau : A \to \mathbb{C}$ satisfying:

- $\tau(1) = 1$;
- $\tau(x^*x) \geq 0$ for all $x \in A$;
- $\tau(xy) = \tau(yx)$ for all $x, y \in A$.

A *tracial von Neumann algebra* is a pair $(A, \tau)$, where $A$ is a von Neumann algebra and $\tau$ is a trace on $A$.

In the case that $\tau$ is also *faithful*, meaning that $\tau(x^*x) = 0 \Rightarrow x = 0$, the function $\langle x, y \rangle_\tau := \tau(y^*x)$ is an inner product on $A$, yielding the so-called *2-norm* $\| \cdot \|_2$ on $A$. The associated metric is complete on any bounded subset of $A$.

$(A, \tau)$ is called *separable* if the metric associated to the 2-norm is separable.
A von Neumann algebra $A$ is said to be a factor if $A \cap A' = \mathbb{C} \cdot 1$.

**Fact**

If $A$ is a von Neumann algebra, then $A \cong \int_X \oplus A_x$ (a direct integral) where each $A_x$ is a factor.

A factor is said to be of type $II_1$ if it is infinite-dimensional and admits a trace.

**Fact**

A $II_1$ factor admits a unique weakly continuous trace, which is automatically faithful.
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A factor is said to be of type $\text{II}_1$ if it is infinite-dimensional and admits a trace.

**Fact**

A $\text{II}_1$ factor admits a unique weakly continuous trace, which is automatically faithful.
Examples-revisited

- $\mathcal{B}(H)$ is a factor. If $\dim(H) < \infty$, then $\mathcal{B}(H)$ admits a trace, but is not a $II_1$ factor. If $\dim(H) = \infty$, then $\mathcal{B}(H)$ admits no trace. Thus, $\mathcal{B}(H)$ is never a $II_1$ factor.

- $L^\infty(X, \mu)$ admits a trace $f \mapsto \int_X f \, d\mu$ but is not a factor.

- If $G$ is a countable group that is ICC, namely all conjugacy classes (other than $\{1\}$) are infinite, then $L(G)$ is a $II_1$ factor; the trace is given by $T \mapsto \langle T\delta_e, \delta_e \rangle$. In particular, if $n \geq 2$, then $L(\mathbb{F}_n)$ is a $II_1$ factor.

- $\mathcal{R}$ is a $II_1$ factor; the trace $\text{tr} : \bigcup_n \otimes_n M_2 \to \mathbb{C}$ extends uniquely to the completion. Moreover, $\mathcal{R}$ embeds into any $II_1$ factor.
Why model theorists care about $\mathcal{II}_1$ factors

- It is straightforward to check that the class of tracial von Neumann algebras (in the correct signature for continuous logic) is a universally axiomatizable class. We let $T_{vNa}$ denote the theory of tracial von Neumann algebras.

- Moreover, it is a fact that the class of $\mathcal{II}_1$ factors is $\forall\exists$-axiomatizable.

- Note that any tracial von Neumann algebra embeds into a $\mathcal{II}_1$ factor: $A \subseteq A \ast L(\mathbb{Z})$ (free product).

- It follows that an existentially closed tracial von Neumann algebra is a $\mathcal{II}_1$ factor.
Suppose that \((A, \tau)\) is a tracial von Neumann algebra and \(\mathcal{U}\) is a nonprincipal ultrafilter on \(\mathbb{N}\). We set

\[
\ell^\infty(A) : = \{(a_n) \in A^\mathbb{N} : \|a_n\| \text{ is bounded}\}.
\]

Unfortunately, if we quotient this out by the ideal

\[
\{(a_n) \in A^\mathbb{N} : \lim_{\mathcal{U}} \|a_n\| = 0\},
\]

the resulting quotient is usually never a von Neumann algebra. Rather, we have to quotient out by the smaller ideal

\[
\{(a_n) \in A^\mathbb{N} : \lim_{\mathcal{U}} \|a_n\|_2 = 0\},
\]

yielding the \textit{tracial ultrapower} \(A^\mathcal{U}\) of \(A\). Continuous logic provided a logical framework for the study of these ultrapowers.
Definition

We say that a separable II$_1$ factor $A$ is $\mathcal{R}^\omega$-embeddable if there is a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $A$ embeds into $\mathcal{R}^\mathcal{U}$.

Remarks

1. If $A$ is $\mathcal{R}^\omega$-embeddable, then $A$ embeds into $\mathcal{R}^\mathcal{U}$ for any nonprincipal ultrafilter on $\mathbb{N}$.

2. $A$ is $\mathcal{R}^\omega$-embeddable if and only if $A \models \text{Th}_\forall(\mathcal{R})$, the universal theory of $\mathcal{R}$. 
In 1976, Connes proved that $L(F_2)$ is $R^\omega$-embeddable.
He then remarked “Apparently such an embedding ought to exist for all $II_1$ factors…”
This remark is now known as the Connes Embedding Problem (CEP) and is the central question in operator algebras. It has zillions of equivalent reformulations.
For example, it is known that $L(G)$ is $R^\omega$-embeddable if and only if $G$ is hyperlinear. So settling the CEP for group von Neumann algebras would settle the question of whether or not all groups are hyperlinear (a serious question in group theory).
Call a separable $II_1$ factor $A$ locally universal if every separable $II_1$ factor is $A^\omega$-embeddable. (So CEP asks whether or not $R$ is locally universal.) Hart, Farah, and Sherman proved the existence of one (and therefore many) locally universal $II_1$ factors (“Poor man’s CEP”).
von Neumann algebras

Model companions

Model complete theories of tracial vNas

Independence relations
Recall that a theory $T$ is *model complete* if any embedding between models of $T$ is elementary.

If $T'$ is a theory, then a model complete theory $T$ is a *model companion* for $T'$ if any model of $T'$ embeds in a model of $T$ and vice-versa (that is, if $T'_\forall = T\forall$). A theory can have at most one model companion.

If $T'$ is universal, then $T'$ has a model companion $T$ if and only if the class of its existentially closed structures is elementary; in this case $T$ is their theory.

**Theorem (G., Hart, Sinclair)**

$T_{vNa}$ does *not* have a model companion.
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**Theorem (G., Hart, Sinclair)**

$T_{vNa}$ does not have a model companion.
Crossed products of tracial vNas

Suppose that $M$ is a von Neumann algebra, $G$ is a countable group and $\alpha : G \to \Aut(M)$ is a group homomorphism. Then there is another von Neumann algebra $M \rtimes_{\alpha} G$ satisfying the following

**Proposition**

1. There is an embedding $I : M \to M \rtimes_{\alpha} G$;
2. $L(G)$ is naturally a subalgebra of $M \rtimes_{\alpha} G$;
3. The action of $G$ on $M$, inside of $M \rtimes_{\alpha} G$, is given by unitary conjugation:
   \[
   I(\alpha_g(x)) = \lambda(h) \circ I(x) \circ \lambda(g^{-1}), \quad x \in M, g \in G.
   \]
4. If $M$ is tracial, then so is $M \rtimes_{\alpha} G$.
5. If $M$ is $\mathcal{R}^\omega$-embeddable and $G$ is amenable, then $M \rtimes_{\alpha} G$ is also $\mathcal{R}^\omega$-embeddable.
\( \mathcal{R} \) does not have QE

**Theorem**

\( \mathcal{R} \) does not have QE.

**Proof.**

- It is enough to find \( \mathcal{R}^\omega \)-embeddable von Neumann algebras \( M \) and \( N \) with \( M \subset N \) and an embedding \( \pi : M \hookrightarrow \mathcal{R}^U \) that does not extend to an embedding \( N \hookrightarrow \mathcal{R}^U \).

- Towards this end, it is enough to find a countable group \( G \) such that \( L(G) \) is \( \mathcal{R}^\omega \)-embeddable, an embedding \( \pi : L(G) \hookrightarrow \mathcal{R}^U \), and \( \alpha \in \text{Aut}(L(G)) \) such that there exists no unitary \( u \in \mathcal{R}^U \) satisfying \( (\pi \circ \alpha)(x) = u\pi(x)u^* \) for all \( x \in L(G) \). (We’ll explain this on the next slide.)

- By nontrivial work of Nate Brown, we can take \( G = \text{SL}(3, \mathbb{Z}) \ast \mathbb{Z} \) and \( \alpha = \text{id} \ast \theta \) for any nontrivial \( \theta \in \text{Aut}(L(\mathbb{Z})) \).
$\mathcal{R}$ does not have QE (cont’d)

Proof.

- Suppose that $G$, $\pi$, and $\alpha$ are as above. Set $M := L(G)$ and $N := M \rtimes_\alpha \mathbb{Z}$. Then $N$ is $\mathcal{R}$-embeddable.
- Suppose, towards a contradiction, that $\pi$ extends to $\tilde{\pi} : N \hookrightarrow \mathcal{R}^U$.
- Let $u \in N$ be the generator of $\mathbb{Z}$ and set $\tilde{u} := \tilde{\pi}(u)$. We then have, for $x \in M$:
  \[
  \tilde{u}\pi(x)\tilde{u}^* = \pi(uxu^*) = \pi(\alpha(x)),
  \]
  contradicting our choice of $\pi$ and $\alpha$. 

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Other non-QE results

Definition
If $A$ is a separable $\text{II}_1$ factor, we say that $A$ is *McDuff* if $A \otimes \mathcal{R} \cong A$.

- For example, $\mathcal{R}$ is McDuff.
- Any $\text{II}_1$ factor $A$ embeds into a McDuff factor: $A \subseteq A \otimes \mathcal{R}$.
- It is a fact that McDuffness is $\forall \exists$-axiomatizable, whence a separable existentially closed tracial von Neumann algebra is a McDuff $\text{II}_1$ factor.

We noticed that Brown’s work would apply if instead of $\mathcal{R}$ we had a locally universal, McDuff $\text{II}_1$ factor. We thus have:

Theorem
*If $S$ is a locally universal, McDuff $\text{II}_1$ factor, then $S$ does not have QE.*
Proof of the Main Theorem

- Suppose, towards a contradiction, that $T_{vNA}$ has a model companion $T$. Since $T_{vNA}$ is $\forall$-axiomatizable and has the amalgamation property, we have that $T$ has QE.

- Fix a separable model $S$ of $T$. As discussed earlier, models of $T$ are then existentially closed tracial von Neumann algebras, whence $S$ is a McDuff $II_1$ factor.

- Moreover, $S$ is a locally universal $II_1$ factor: if $A$ is an arbitrary separable tracial vNA, then $A$ embeds in some separable $S_1 \models T$. Since $S^u$ is $\omega_1$-saturated, $S_1$ embeds in $S^u$, whence $A$ embeds in $S^u$.

- By our previous theorem, $S$ does not have QE, a contradiction.  $\square$
1 von Neumann algebras

2 Model companions

3 Model complete theories of tracial vNas

4 Independence relations
Are there model complete theories of tracial vNas?

Just because there is no model companion of $T_{vNA}$ does not prevent there from being a model-complete theory of tracial von Neumann algebras, so we raise the question: Is there a model-complete theory of tracial von Neumann algebras (whose models would automatically be $II_1$ factors)?

Theorem (G., Hart, Sinclair)

If the CEP has a positive solution, then there is no model-complete theory of tracial von Neumann algebras.
Are there model complete theories of tracial vNas?

Just because there is no model companion of $T_{vNA}$ does not prevent there from being a model-complete theory of tracial von Neumann algebras, so we raise the question: Is there a model-complete theory of tracial von Neumann algebras (whose models would automatically be $II_1$ factors)?

**Theorem (G., Hart, Sinclair)**

*If the CEP has a positive solution, then there is no model-complete theory of tracial von Neumann algebras.*
A preliminary result

Fact (Jung)

Any embedding $\mathcal{R} \to \mathcal{R}^U$ is unitarily equivalent to the diagonal embedding (whence elementary).

Remark

Jung’s result shows that $\mathcal{R}$ is the prime model of its theory.
**Proposition**

Suppose that $A$ is an $\mathcal{R}^\omega$-embeddable $II_1$ factor such that $\text{Th}(A)$ is model-complete. Then $A \equiv \mathcal{R}$.

**Proof.**

Draw crude diagram on the board.

We now see how CEP implies that there is no model-complete theory of $II_1$ factors. Indeed, if $T$ were a model-complete theory of $II_1$ factors, then by CEP, models of $T$ would be $\mathcal{R}^\omega$-embeddable, whence the above proposition shows $T = \text{Th}(\mathcal{R})$. Another use of CEP shows that $T_\forall = T_{vNa}$, whence $T$ is a model companion for $T_{vNa}$, which we know has no model companion.
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Free Group Factors

- Murray and von Neumann showed that $L(\mathbb{F}_n) \not\cong \mathcal{R}$ by showing that $\mathcal{R}$ has a certain property, called $(\Gamma)$, that $L(\mathbb{F}_n)$ does not have.

- It is not too difficult to show that $(\Gamma)$ is axiomatizable by a set of sentences in continuous logic, whence $L(\mathbb{F}_n) \not\cong \mathcal{R}$.

- Since $L(\mathbb{F}_n)$ is $\mathcal{R}^\omega$-embeddable, we see that $\text{Th}(\mathbb{F}_n)$ is not model-complete.

- **Big Open Question:** For distinct $m, n \geq 2$, is $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$?

- **Weaker, but still difficult, Open Question:** For distinct $m, n \geq 2$, is $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$?

- If the above question has an affirmative answer, we see that this common theory is not model-complete. But are the natural embeddings $L(\mathbb{F}_m) \hookrightarrow L(\mathbb{F}_n)$ (for $m < n$) elementary (like in the case of $\text{Th}(\mathbb{F}_n)$)?
1. von Neumann algebras

2. Model companions

3. Model complete theories of tracial vNas

4. Independence relations
The order property

Definition

Suppose that $M$ is a metric structure, $\varphi(x; y)$ is a formula with $|x| = |y| = n$, and $\epsilon > 0$.

- For $a, b \in M^n$, we write $a \prec_{\varphi, \epsilon} b$ if $\varphi(a, b) \leq \epsilon$ and $\varphi(b, a) \geq 1 - \epsilon$.

- A $\varphi$-$\epsilon$ chain of length $k$ in $M$ is a sequence $a_1, \ldots, a_k$ from $M^n$ such that $a_i \prec_{\varphi, \epsilon} a_j$ for $1 \leq i < j \leq k$.

- $M$ has the order property or is unstable if there exists $\varphi$ such that, for every $\epsilon > 0$, $M$ has arbitrarily long finite $\varphi$-$\epsilon$ chains.

- A sequence $(M_i : i \in \mathbb{N})$ of structures has the order property if there exists $\varphi$ such that, for every $\epsilon > 0$ and every $k \in \mathbb{N}$, all but finitely many of the $M_i$ have a $\varphi$-$\epsilon$ chain of length $k$. 
**The order property**

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Independence relations

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- For $a, b \in M^n$, we write $a \preceq_{\varphi, \epsilon} b$ if $\varphi(a, b) \leq \epsilon$ and $\varphi(b, a) \geq 1 - \epsilon$.
- A *$\varphi$-$\epsilon$ chain of length $k$ in $M$* is a sequence $a_1, \ldots, a_k$ from $M^n$ such that $a_i \preceq_{\varphi, \epsilon} a_j$ for $1 \leq i < j \leq k$.
- $M$ has the *order property* or is *unstable* if there exists $\varphi$ such that, for every $\epsilon > 0$, $M$ has arbitrarily long finite $\varphi$-$\epsilon$ chains.
- A sequence $(M_i : i \in \mathbb{N})$ of structures has the order property if there exists $\varphi$ such that, for every $\epsilon > 0$ and every $k \in \mathbb{N}$, all but finitely many of the $M_i$ have a $\varphi$-$\epsilon$ chain of length $k$. 

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Tracial vNas don’t have a model companion

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The order property

Definition

Suppose that $M$ is a metric structure, $\varphi(x; y)$ is a formula with $|x| = |y| = n$, and $\epsilon > 0$.

- For $a, b \in M^n$, we write $a \prec_{\varphi, \epsilon} b$ if $\varphi(a, b) \leq \epsilon$ and $\varphi(b, a) \geq 1 - \epsilon$.
- A $\varphi$-$\epsilon$ chain of length $k$ in $M$ is a sequence $a_1, \ldots, a_k$ from $M^n$ such that $a_i \prec_{\varphi, \epsilon} a_j$ for $1 \leq i < j \leq k$.
- $M$ has the order property or is unstable if there exists $\varphi$ such that, for every $\epsilon > 0$, $M$ has arbitrarily long finite $\varphi$-$\epsilon$ chains.
- A sequence $(M_i : i \in \mathbb{N})$ of structures has the order property if there exists $\varphi$ such that, for every $\epsilon > 0$ and every $k \in \mathbb{N}$, all but finitely many of the $M_i$ have a $\varphi$-$\epsilon$ chain of length $k$. 
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The order property in matrix algebras

Theorem (Hart. Farah, Sherman)

The sequence \((M_{2n} : n \in \mathbb{N})\) has the order property.

Proof.

Let \(x = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\) and for \(1 \leq i \leq n - 1\), let

\[
a_i = \bigotimes_{j=0}^{i} x \otimes \bigotimes_{j=i+1}^{n-1} 1 \quad \text{and} \quad b_i = \bigotimes_{j=0}^{i} 1 \otimes x^* \otimes \bigotimes_{j=i+2}^{n-1} 1.
\]

Set \(\varphi(x_1, x_2; y_1 y_2) := \| [x_1, y_2] \|_2\) and observe that, for \(i < j\), we have \(\varphi(a_i, b_i; a_j, b_j) = 0\) and \(\varphi(a_j, b_j; a_i, b_i) = 2\). \(\square\)
**Independence relations**

**II₁ factors are unstable**

**Corollary (Hart, Farah, Sherman)**

*Every II₁ factor has the order property.*

**Proof.**

Every II₁ factor contains a copy of $M_{2^n}$.

**Corollary (Hart, Farah, Sherman)**

*Assuming ($\neg \text{CH}$), any separable II₁ factor has two nonisomorphic ultrapowers.*

**Folkloric Theorem (Hart)**

*Any II₁ factor is not (model-theoretically) simple.*
Rosiness?

We had hoped that, although $II_1$ factors are not even simple, perhaps there could be a well-behaved notion of independence. Here is a natural candidate:

- Suppose that $M$ is a $II_1$ factor. Let $L^2 M$ be the completion of the inner product space $(M, \langle \cdot, \cdot \rangle_\tau)$.
- For a subalgebra $N$ of $M$, we let $E_N : L^2 M \to L^2 N$ be the orthogonal projection map ("conditional expectation").
- For $D \subseteq M$, let $\langle D \rangle$ denote the von Neumann subalgebra of $M$ generated by $D$.
- Define $A \downarrow_C B$ to hold if and only if, for all $a \in A$, we have $E_{\langle BC \rangle}(a) = E_{\langle C \rangle}(a)$.
- We had an idea how to prove that $\downarrow$ is an independence relation for $\text{Th}(R)$ assuming QE. Without QE, this seems very difficult.
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- Define $A \underset{\mathcal{C}}{\subseteq} B$ to hold if and only if, for all $a \in A$, we have $E_{\langle BC \rangle}(a) = E_{\langle C \rangle}(a)$.
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Independence relations

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- For $D \subseteq M$, let $\langle D \rangle$ denote the von Neumann subalgebra of $M$ generated by $D$.
- Define $A \downarrow_C B$ to hold if and only if, for all $a \in A$, we have $E_{\langle BC \rangle}(a) = E_{\langle C \rangle}(a)$.
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2. For a subalgebra $N$ of $M$, we let $E_N : L^2 M \to L^2 N$ be the orthogonal projection map ("conditional expectation").

3. For $D \subseteq M$, let $\langle D \rangle$ denote the von Neumann subalgebra of $M$ generated by $D$.

4. Define $A \downarrow_{\mathbb{C}} B$ to hold if and only if, for all $a \in A$, we have $E_{\langle B \rangle}(a) = E_{\langle C \rangle}(a)$.

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- We had an idea how to prove that $\downarrow_C$ is an independence relation for Th($\mathcal{R}$) assuming QE. Without QE, this seems very difficult.
References

