Chern classes of singular varieties, graph hypersurfaces, and Feynman integrals

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LIB60BER

Joint work with Matilde Marcolli.

Main references:

- Feynman motives of banana graphs. Comm. in Number Theory and Physics (2009) 1-57
- Algebro-Geometric Feynman rules. arXiv:0811.2514
- Parametric Feynman integrals and determinant hypersurfaces. arXiv:0901.2107
- Matilde Marcolli: Feynman Motives. World Scientific. (To appear later this year.)

Perturbative QFT $\rightsquigarrow$ Feynman integral computations
Extensive numerical evidence: graph 'amplitudes' are linear combinations of multiple zeta values (Broadhurst-Kreimer).
Hard to give a precise statement, as integrals typically diverge.
Γ: graph; p: 'momenta' attached to external edges

$$U(\Gamma,p)=\frac{\Gamma(n-D\ell/2)}{(4\pi)^{\ell D/2}}\int_{[0,1]^n}\frac{\delta(1-\sum_i t_i)V_\Gamma(t,p)^{D\ell/2-n}}{\Psi_\Gamma(t)^{D/2}}\,dt_1\cdots dt_n.$$

- $n = \#$ internal edges
- $D =$ spacetime dimension
- $\ell = b_1(\Gamma) = \#$ loops
- $V_\Gamma(t,p) =$ a rational function
- $\psi_\Gamma(t)=$ a polynomial of degree ℓ determined by the graph.

Ignore most of this!

$U(\Gamma, p)$ = an integral of a form defined over the complement of a hypersurface X_Γ: $\{\psi_\Gamma = 0\}$ in projective space.

X_Γ is determined by the graph Γ, in a way that I will explain later.

There are *renormalization* techniques assigning well-defined values to such (typically divergent) integrals. These are beyond my understanding, but their success is unquestionable.

Broadhurst-Kreimer $\rightsquigarrow$ evidence that numbers obtained this way are periods of 'mixed Tate motives'.

Mixed Tate motives, simple-minded viewpoint:
Varieties admitting decompositions as unions, set-differences of affine spaces determine Tate motives.

Mixed Tate motive: an object in the smallest motivic category generated by Tate motives.

Motive: in this talk, will approximate these by elements of the Grothendieck ring of varieties.

Grothendieck ring of varieties: a Lego construction set, with bricks given by isomorphism class of varieties.
Addition $\leftrightarrow$ disjoint union; Multiplication $\leftrightarrow$ product.

This gives a 'universal Euler characteristic': e.g., $X \rightsquigarrow \chi(X)$ (top. Euler characteristic) factors through the Grothendieck ring.

'Mixed Tate motives': Use only affine spaces as Lego bricks.

Examples:

$[\mathbb{P}^n] = [\mathbb{A}^n] + [\mathbb{A}^{n-1}] + \cdots + [\mathbb{A}^0]$.

Grassmannians, Schubert varieties...

Blow-up of $\mathbb{P}^n$ along $\mathbb{P}^m$: $[\mathbb{P}^n] - [\mathbb{P}^m] + [\mathbb{P}^m] \cdot [\mathbb{P}^{n-m-1}]$.

Caveat: intersections of Tate motives are not necessarily Tate motives.

Broadhurst-Kreimer: evidence that contributions of individual graphs to Feynman integrals are periods of mixed Tate motives.
Kontsevich: BK evidence may be explained if the motives determined by graph hypersurfaces X_Γ are mixed-Tate motives.

Belkale-Brosnan: not true. Graph hypersurfaces generate the Grothendieck ring of varieties! (But the proof is non-constructive.)

Program:

- Analyze classes of graphs, attempt to estimate 'complexity' of X_Γ in the Grothendieck ring.
- Note: A hypersurface in $\mathbb{P}^n$ can be 'simple' in Grothendieck ring only if it is 'very' singular.
- 'Quantify' singularity: compute Milnor classes of graph hypersurfaces. (Milnor $= c_{\rm F} - c_{\rm SM}$.)
- Tools needed to compute the $c_{\rm SM}$ class of X_Γ usually suffice in order to compute class in Grothendieck ring.

Is this really the right approach?

(After Abraham Kaplan, *The conduct of inquiry*, 1964)

There is a story of a drunkard searching under a street lamp for his house key, which he had dropped some distance away.

Asked why he didn't look where he had dropped it, he replied, "It's lighter here!"

There is a Lego-like theory of characteristic classes for possibly singular varieties in characteristic 0 (say: over $\mathbb{C}$).

History:

- Marie-Hélène Schwartz
  ($\sim$1964, Poincaré-Hopf for singular varieties);
- Grothendieck-Deligne
  ($\sim$1969, SGA5; conjectural 'functorial' theory);
- Robert MacPherson
  ($\sim$1974, affermative answer to Grothendieck-Deligne);
- Brasselet-Schwartz
  ($\sim$1979, Schwartz=MacPherson).

$\rightsquigarrow$ Chern-Schwartz-MacPherson ($c_{\rm SM}$) classes of compact complex algebraic varieties.

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'Normalization': X nonsingular $\rightsquigarrow c_{\rm SM}(X) = c(TX) \cap [X]$.

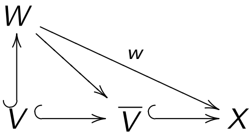
'Functoriality': for any X, $c_{\rm SM}(X) = c_*(1\!\!1_X)$, where c_* is a *natural transformation* from the functor of constructible functions to the Chow ('homology') functor, w.r.t. proper morphisms.

First instance of functoriality: $\int c_{\rm SM}(X) = \chi(X)$ (topological Euler characteristic).
'Singular Poincaré-Hopf'

MacPherson: explicit construction of this natural transformation.

Definition (Warning: not à la Schwartz, nor à la MacPherson.)

Write $X = \amalg_{i=1}^{n} V_i$, for V_i nonsingular (of course, possibly noncompact). I will define a contribution $c_*(1\!\!1_V) \in A_* X$ for each nonsingular $V \subseteq X$.



- $W :=$ resolution of singularities of $\overline{V}$.
- $D := W \smallsetminus V$, assume divisor with SNC.

Definition

$$c_*(1\!\!1_V) := w_*(c(\Omega^1_W(\log D)^\vee) \cap [W])$$

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Write $X = \amalg_{i=1}^n V_i$, for V_i nonsingular, in any way.

Definition: Chern-Schwartz-MacPherson class

$c_{\rm SM}(X) := \sum_i c_*(1\!\!1_{V_i})$

(Clearly Lego-like!)

Theorem (—, 2006)

This is independent of all choices, and agrees with Schwartz/MacPherson's definition.

Two proofs:

- Using MacPherson's result, easy exercise.

Classes de Chern pour variétés singulières, revisitées,
C. R. Math. Acad. Sci. Paris 342 (2006), no. 6, 405–410.

- Not using MacPherson's natural transformation, prove directly that c_* satisfies the Grothendieck-Deligne conjecture.

Limits of Chow groups and a new construction of Chern-Schwartz-MacPherson classes, Pure Appl. Math. Q. (2006) (MacPherson Volume 2), 915–941.

Useful side-product: functoriality with respect to not necessarily proper morphisms, for an 'enlarged' Chow functor.

X: a subscheme of a nonsingular variety M.

Definition: Chern-Fulton class

$c_{\mathrm{F}}(X) := c(TM) \cap s(X, M)$

Example: X a hypersurface in M, then

$$c_{\mathrm{F}}(X) := c(TM) \cap (c(N_X M)^{-1} \cap [X]) = c(TM) \cap \frac{[X]}{1+X} \quad .$$

Possibly better name for this: 'virtual Chern class' of X.
If X is nonsingular, $c_{\mathrm{F}}(X) = c(TX) \cap [X]$.
Morally, $c_{\mathrm{F}}(X)$ is 'the Chern class of a smoothening of X'.
A precise statement of this type: Fantechi-Göttsche 2007, Theorem 4.15.

Remark: the virtual class is not Lego-like.
In particular, $c_{\mathrm{F}}(X) \neq c_{\mathrm{SM}}(X)$ in general.
Link between $c_{\mathrm{F}}(X)$, $c_{\mathrm{SM}}(X)$: reasonably well-understood for hypersurfaces, complete intersections. (Work of many people.)

Yokura: The difference is captured by Verdier-Riemann-Roch-type results. (Close to Grothendieck's motivation!)

Definition: Milnor class (up to sign...)

$c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X)$

- If X is nonsingular, Milnor class $= 0$.
- If X is a hypersurface, then $\pm \int c_{\mathrm{F}}(X) - c_{\mathrm{SM}}(X) =$ sum of Parusiǹski-Milnor numbers of singularities. (Hence the name.)
- In general, a quantification of 'how singular X is'.

Recall from 10 minutes ago:
The aim is to study 'graph hypersurfaces', in the Grothendieck group and from the point of view of singularities.

Γ: graph; one variable t_e for each edge e
(Usually assume Γ is connected and 1–PI: it cannot be disconnected by removing a single edge.)

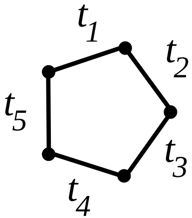
Definition

$$\Psi_\Gamma(t) = \sum_{T \subseteq \Gamma} \prod_{e \notin E(T)} t_e$$

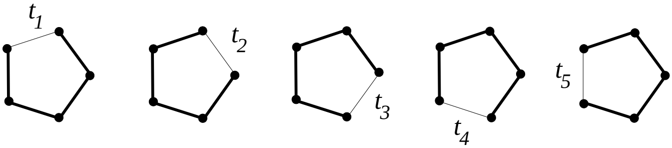
where the sum is over all the spanning trees T of Γ.

$\#$ of variables $= \#$ (internal) edges; degree $= \#$ loops

Example: Γ=n-sided polygon

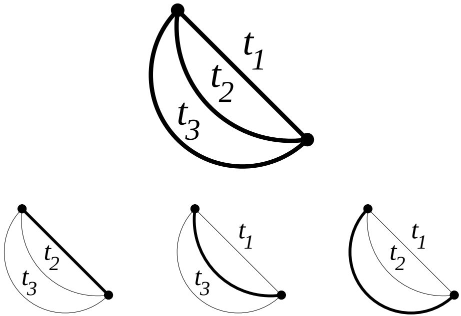


List all spanning trees, and edges *missed* by the spanning trees:



$\rightsquigarrow \psi_\Gamma = t_1 + t_2 + \cdots + t_n$.

Example: 'banana graphs': two vertices, n parallel edges



$\rightsquigarrow \psi_\Gamma = t_2 t_3 + t_1 t_3 + t_1 t_2$ for $n = 3$.

$$\psi_\Gamma = t_1 \cdots t_n \left(\frac{1}{t_1} + \cdots + \frac{1}{t_n} \right)$$

$\{\psi_\Gamma = 0\}$: hypersurface $X_\Gamma \subseteq \mathbb{P}^{n-1}$ (or $\hat{X}_\Gamma \subseteq \mathbb{A}^n$); $\deg X_\Gamma = \ell$.
(n = number of edges of Γ; ℓ = number of loops.)

Task: compute the class $[X_\Gamma]$ in the Grothendieck ring, and/or $c_{\mathrm{SM}}(X_\Gamma) \in A_* \mathbb{P}^{n-1}$.

Equivalent: $[\mathbb{P}^{n-1} \smallsetminus X_\Gamma]$, $c_{\mathrm{SM}}(1\!\!1_{\mathbb{P}^{n-1} \smallsetminus X_\Gamma}) \in A_* \mathbb{P}^{n-1}$.
(Closer to motivation: the Feynman amplitude of Γ is a period of the *complement* of X_Γ.)

For example: $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma) = ?$

Relation between these invariants and combinatorics of Γ?

For instance, does $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$ closely reflect the combinatorics of Γ?

Devil's advocate (=referee to CNTP 2009): maybe not too closely. Indeed, $\chi(\mathbb{P}^N \smallsetminus X_{\Gamma_1 \amalg \Gamma_2}) = 0$. (Reason: $\mathbb{C}^*$-action.)

Challenge: Beyond computing invariants for individual graphs, understand the organization of these invariants for all graphs. (This is in fact necessary in order to approach renormalization issues.)

In low dimension, $c_{\rm SM}$ classes may be computed with e.g. Macaulay2.
http://www.math.fsu.edu/~aluffi/CSM/CSMexamples.html

Experimentation for small graphs: J. Stryker, almost all graphs with six or fewer edges.

Puzzle: $c_{\rm SM}(X_\Gamma)$ is *effective* for all these graphs! Why?

Evokes:

- $c_{\rm SM}(T)$ is effective for all toric varieties. ("Ehlers' formula")
- $c_{\rm SM}(S)$ is conjecturally effective for all Schubert varieties S of ordinary Grassmannians (— & Mihalcea, JAG 2009)

Note these are all mixed-Tate. . .

Infinite families of graphs?

Theorem (—, Marcolli, CNTP 2009)

Explicit computation of $[X_\Gamma] \in$ Grothendieck ring, and $c_{\mathrm{SM}}(X_\Gamma)$, for $\Gamma =$ all banana graphs

In the Grothendieck group:

$$[\mathbb{P}^{n-1} \smallsetminus X_{\Gamma_n}] = \frac{\mathbb{T}^n - (-1)^n}{\mathbb{T}+1} + n\,\mathbb{T}^{n-2}$$

$$= \mathbb{T}^{n-1} + (n-1)\mathbb{T}^{n-2} + \mathbb{T}^{n-3} - \mathbb{T}^{n-4} + \mathbb{T}^{n-5} + \cdots \pm 1$$

where $\mathbb{T} = [\mathbb{A}^1 \smallsetminus \mathbb{A}^0] = \mathbb{L} - 1$.
$(\mathbb{L} = [\mathbb{A}^1])$

The CSM class:

$$c_{\mathrm{SM}}(1\!\!1_{\mathbb{P}^{n-1}\smallsetminus X_{\Gamma_n}}) = ((1-H)^{n-1} + nH) \cap [\mathbb{P}^{n-1}]$$

where H is the hyperplane class in $\mathbb{P}^{n-1}$.

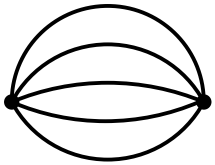
'Large' Milnor class ($\leftrightarrow$'very singular'). Example, $n = 9$:
$84H^3 - 1176H^4 + 9786H^5 - 78792H^6 + 630516H^7 - 5044200H^8$

Corollary: $\chi(X_\Gamma) = n + (-1)^n$ for $n \geq 3$.

In particular, $\chi(X_\Gamma) > 0$ for all banana graphs.
In fact, $c_{\mathrm{SM}}(X_\Gamma)$ is effective for banana graphs.

Proof of the theorem:

If Γ is any planar graph, can relate X_Γ to $X_{\Gamma^\vee}$, where $\Gamma^\vee$ is the dual graph: they correspond to each other via a Cremona transformation of $\mathbb{P}^{n-1}$.
For $\Gamma =$ banana graphs:



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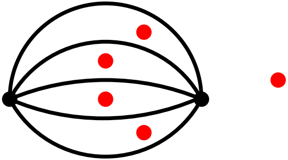

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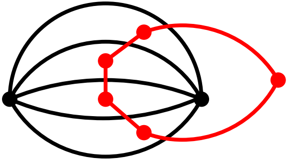
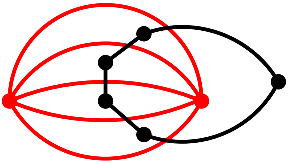




$\Gamma^\vee$ are polygons, computation can be carried out explicitly. (Calculus of constructible functions, and lemma on $c_{\rm SM}$ classes via 'adapted blow-ups'.) □

Remark: More generally, one expects certain sums of $[X_\Gamma]$ to be 'easier' (and more interesting) than individual $[X_\Gamma]$.
Bloch, 2008: computation of $\sum [X_\Gamma]$, Γ connected graph with N vertices (with automorphism factor); it is MT. Main tool: the relation between $[X_\Gamma]$ and $[X_{\Gamma^\vee}]$, extended to non-planar graphs.

Reason why Γ assumed to be connected, 1–PI:
Integrals $U(\Gamma, p)$ are multiplicative on disjoint unions of graphs. If $\Gamma = \Gamma_1 \amalg \Gamma_2$, then

$$U(\Gamma, p) = U(\Gamma_1, p_1) U(\Gamma_2, p_2)$$

If Γ is obtained by joining Γ_1, Γ_2 by an edge (matching external momenta), multiply product by a 'propagator' term.

FEYNMAN RULES!

With Marcolli: *'Algebro-geometric Feynman rules'*
(I vetoed 'Feynman rules in algebraic geometry')

Back to the challenges presented earlier:

Challenge: Understand the organization of invariants such as $[\mathbb{P}^{n-1} \smallsetminus X_\Gamma]$, $c_{\mathrm{SM}}(1\!\!1_{\mathbb{P}^{n-1} \smallsetminus X_\Gamma})$ for all graphs. Understand relation between the combinatorics of a graph and the corresponding invariants.

Ways to formalize these:

- Give formulas for the behavior of invariants after combinatorial operations such as splitting edges, adding edges...
- Look for 'Feynman rules' based on the class in the Grothendieck ring and on c_{SM} classes.

First task: some formulas are obtained in CNTP 2009.
Second task: maybe more interesting.

The following recipe is part of a larger picture:

- Γ: finite graph (may be non-connected, non-1-PI...), n edges
- $\hat{X}_\Gamma$: corr. hypersurface in $\mathbb{A}^n$; view as locally closed in $\mathbb{P}^n$
- $c_*(1\!\!1_{\hat{X}_\Gamma}) = a_0[\mathbb{P}^0] + \cdots + a_n[\mathbb{P}^n]$
- Define $G_\Gamma(T) = a_0 + a_1 T + \cdots + a_n T^n$
- Define $C_\Gamma(T) = (T+1)^n - G_\Gamma(T)$

Example: Γ = banana graph $\rightsquigarrow C_\Gamma(T) = T(T-1)^{n-1} + nT^{n-1}$
Remarks:

- Coefficient of T^{n-1} in $C_\Gamma(T)$ equals $n - \ell$.
- $C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$.

Theorem (—, Marcolli, arXiv:0811.2514)

The invariant $C_\Gamma(T)$ obeys the Feynman rules, with inverse propagator $(T+1)$.

Proof:
Show that Feynman rules correspond to homomorphisms from a 'Grothendieck ring' of conical immersed subvarieties of $\mathbb{A}^n$.
The function $G_\Gamma(T)$ is such a homomorphism.
Proof of *this* fact: study $c_{\rm SM}$ classes of joins in projective space. □

$\leadsto$ 'Feynman rules' for $c_{\rm SM}$ classes of graph hypersurfaces are a particular case of behavior of $c_{\rm SM}$ classes with respect to natural constructions in projective geometry.

Note that this answers the objection on $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$:
This is one coefficient of $C_\Gamma(T)$; it is not multiplicative under disjoint union, but $C_\Gamma(T)$ is.

Similar story at the level of motives:

$\Gamma \rightsquigarrow [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma]$.

Theorem (—, Marcolli, arXiv:0811.2514)

This invariant also satisfies the Feynman rules, with inverse propagator $\mathbb{L} = [\mathbb{A}^1]$.

In arXiv:0811.2514, we obtain a 'universal' invariant.

More recent work with Marcolli: a possible approach to explaining the BK evidence. (Reference: arXiv:0901.2107.)

Idea: Transfer the integral computation to a fixed variety D_ℓ (for given number ℓ of loops) $\rightsquigarrow$ for all graphs with ℓ loops, the Feynman integral is a period of a fixed D_ℓ relative to a locus S_ℓ supported on strata of a fixed normal crossing divisor.

Here, D_ℓ is the complement of the determinant hypersurface, clearly MT.

The translation holds for graphs satisfying reasonable combinatorial conditions, e.g.: 3-*vertex connected, each vertex admits a wheel neighborhood.*

This reduces the question to 'linear algebra': describe a variety of frames $(v_1, \ldots, v_\ell)$ with $v_1 \in V_1$, ..., $v_\ell \in V_\ell$, where $V_1, \ldots V_\ell$ are (arbitrary) subspaces of a fixed vector space.

Prove this is MT!

Ravi Vakil: This is bound to be hard.
('Murphy's law in algebraic geometry')

Low ℓ (=few loops): fun exercise.

Example: V_1, V_2: arbitrary subspaces of a fixed vector space V;
$\mathbb{F}(V_1, V_2) =$ variety of pairs (v_1, v_2) s.t. $v_i \in V_i$, and (v_1, v_2) linearly independent.

$[\mathbb{F}(V_1, V_2)] =??$

$d_i = \dim V_i$; $d_{12} = \dim(V_1 \cap V_2)$:

$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$

$\ell = 3$, notation as above ($\delta = \dim(V_1 + V_2 + V_3)$):

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$
$$-(\mathbb{L} - 1)\left((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1)\right)$$
$$+(\mathbb{L} - 1)^2\left(\mathbb{L}^{d_1+d_2+d_3-\delta} - \mathbb{L}^{d_{123}+1}\right) + (\mathbb{L} - 1)^3$$

In particular, both are mixed-Tate. (Both from arXiv:0901.2107.)

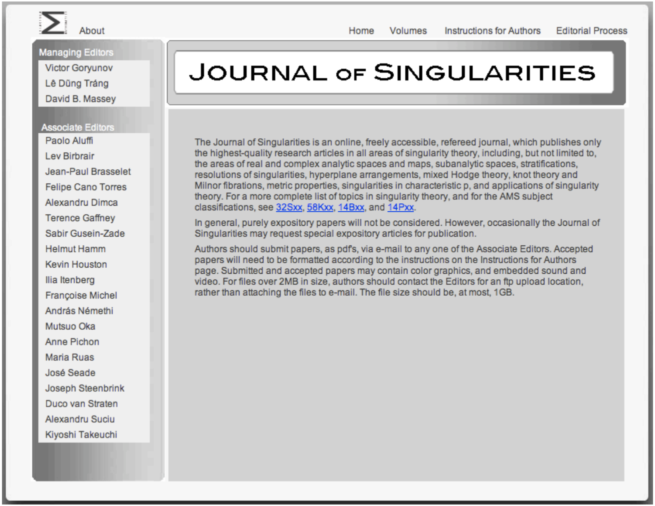
$\ell = 4$: some work by J. Fullwood; but it gets very messy very fast.

$\mathbb{F}(V_1, \dots, V_r)$ may be expressed as an intersection of Schubert varieties in flag manifolds; these tend to be very complex gadgets. (And remember: intersections of MT are not necessarily MT!)

SUMMARY:

- Numerical evidence suggests that individual contributions of graphs to Feynman integrals may be ‘very special’ numbers.
- One way to approach this question is to study certain (very) singular varieties associated to graphs.
- Classes in the Grothendieck group and characteristic classes are natural ways to quantify ‘how singular’ these varieties are.
- It turns out that these invariants satisfy the ‘Feynman rules’, a natural set of constraints in the theory of Feynman integrals.
- A new approach reduces the question to the study of certain varieties of frames, with relations to e.g. the geometry of Schubert varieties in flag manifolds.

Just two more things. . .



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