Chern classes of singular varieties, graph hypersurfaces, and Feynman integrals

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Jaca, June 26, 2009

LIB60BER
1 Introduction

2 Crash course on Chern/Milnor classes of singular varieties
   - Chern-Schwartz-MacPherson classes
   - Chern-Fulton class
   - Milnor class

3 Invariants of graph hypersurfaces
   - Definition
   - Explicit computations
   - Feynman rules
   - Back to Broadhurst-Kreimer

4 Two more things
Joint work with Matilde Marcolli.
Main references:

- **Feynman motives of banana graphs.** Comm. in Number Theory and Physics (2009) 1-57
- **Algebro-Geometric Feynman rules.** arXiv:0811.2514
- **Parametric Feynman integrals and determinant hypersurfaces.** arXiv:0901.2107
- **Matilde Marcolli:** *Feynman Motives.* World Scientific. (To appear later this year.)
Perturbative QFT $\leadsto$ Feynman integral computations

Extensive numerical evidence: graph ‘amplitudes’ are linear combinations of multiple zeta values (Broadhurst-Kreimer).

Hard to give a precise statement, as integrals typically diverge.

$\Gamma$: graph; $p$: ‘momenta’ attached to external edges

$$U(\Gamma, p) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{D/2}} \int_{[0,1]^n} \frac{\delta(1 - \sum_i t_i)V_\Gamma(t, p)^{D\ell/2-n}}{\Psi_\Gamma(t)^{D/2}} dt_1 \cdots dt_n.$$ 

- $n = \#$ internal edges
- $D =$ spacetime dimension
- $\ell = b_1(\Gamma) =$ $\#$ loops
- $V_\Gamma(t, p) =$ a rational function
- $\psi_\Gamma(t) =$ a polynomial of degree $\ell$ determined by the graph.
Ignore most of this!

\[ U(\Gamma, p) = \text{an integral of a form defined over the complement of a hypersurface } X_\Gamma: \{\psi_\Gamma = 0\} \text{ in projective space.} \]

\(X_\Gamma\) is determined by the graph \(\Gamma\), in a way that I will explain later.

There are renormalization techniques assigning well-defined values to such (typically divergent) integrals. These are beyond my understanding, but their success is unquestionable.

Broadhurst-Kreimer ⇝ evidence that numbers obtained this way are periods of ‘mixed Tate motives’.
Mixed Tate motives, simple-minded viewpoint:
Varieties admitting decompositions as unions, set-differences of
affine spaces determine Tate motives.

Mixed Tate motive: an object in the smallest motivic category
generated by Tate motives.

Motive: in this talk, will approximate these by elements of the
Grothendieck ring of varieties.

Grothendieck ring of varieties: a Lego construction set, with bricks
given by isomorphism class of varieties.
Addition $\leftrightarrow$ disjoint union; Multiplication $\leftrightarrow$ product.

This gives a ‘universal Euler characteristic’: e.g., $X \leadsto \chi(X)$
(top. Euler characteristic) factors through the Grothendieck ring.
‘Mixed Tate motives’: Use only affine spaces as Lego bricks.

Examples:

\[ [\mathbb{P}^n] = [\mathbb{A}^n] + [\mathbb{A}^{n-1}] + \cdots + [\mathbb{A}^0]. \]

Grassmannians, Schubert varieties.

Blow-up of \( \mathbb{P}^n \) along \( \mathbb{P}^m \): \( [\mathbb{P}^n] - [\mathbb{P}^m] + [\mathbb{P}^m] \cdot [\mathbb{P}^{n-m-1}]. \)

Caveat: intersections of Tate motives are not necessarily Tate motives.
Broadhurst-Kreimer: evidence that contributions of individual graphs to Feynman integrals are periods of mixed Tate motives. Kontsevich: BK evidence may be explained if the motives determined by graph hypersurfaces $X_{\Gamma}$ are mixed-Tate motives. Belkale-Brosnan: not true. Graph hypersurfaces generate the Grothendieck ring of varieties! (But the proof is non-constructive.)

Program:

- Analyze classes of graphs, attempt to estimate ‘complexity’ of $X_{\Gamma}$ in the Grothendieck ring.
- Note: A hypersurface in $\mathbb{P}^n$ can be ‘simple’ in Grothendieck ring only if it is ‘very’ singular.
- ‘Quantify’ singularity: compute Milnor classes of graph hypersurfaces. ($\text{Milnor} = c_F - c_{SM}$.)
- Tools needed to compute the $c_{SM}$ class of $X_{\Gamma}$ usually suffice in order to compute class in Grothendieck ring.
Is this really the right approach?

(After Abraham Kaplan, *The conduct of inquiry*, 1964)

There is a story of a drunkard searching under a street lamp for his house key, which he had dropped some distance away.

Asked why he didn’t look where he had dropped it, he replied, “It’s lighter here!”
There is a Lego-like theory of characteristic classes for possibly singular varieties in characteristic 0 (say: over \( \mathbb{C} \)).

**History:**

- Marie-Hélène Schwartz
  (~1964, Poincaré-Hopf for singular varieties);
- Grothendieck-Deligne
  (~1969, SGA5; conjectural ‘functorial’ theory);
- Robert MacPherson
  (~1974, affirmative answer to Grothendieck-Deligne);
- Brasselet-Schwartz
  (~1979, Schwartz=MacPherson).

\( \rightsquigarrow \) **Chern-Schwartz-MacPherson \((c_{SM})\)** classes of compact complex algebraic varieties.
Chern-Schwartz-MacPherson \((c_{\text{SM}})\) classes of compact complex algebraic varieties.

‘Normalization’: \(X\) nonsingular \(\leadsto c_{\text{SM}}(X) = c(TX) \cap \llbracket X \rrbracket\).

‘Functoriality’: for any \(X\), \(c_{\text{SM}}(X) = c_\ast(\mathbb{1}_X)\), where \(c_\ast\) is a natural transformation from the functor of constructible functions to the Chow (‘homology’) functor, w.r.t. proper morphisms.

First instance of functoriality: \(\int c_{\text{SM}}(X) = \chi(X)\)
(topological Euler characteristic).

‘Singular Poincaré-Hopf’

MacPherson: explicit construction of this natural transformation.
**Definition** (Warning: not à la Schwartz, nor à la MacPherson.)

Write $X = \amalg_{i=1}^n V_i$, for $V_i$ nonsingular (of course, possibly noncompact). I will define a contribution $c_*(\mathbb{1}_V) \in A_*X$ for each nonsingular $V \subseteq X$.

\begin{align*}
W & := \text{resolution of singularities of } \overline{V}. \\
D & := W \setminus V, \text{ assume divisor with SNC.}
\end{align*}

**Definition**

\[ c_*(\mathbb{1}_V) := w_*(c(\Omega^1_W(\log D)\wedge) \cap [W]) \]
Definition

\[ c_*(\mathbb{1}_V) := w_*(c(\Omega^1_W(\log D)^c) \cap [W]) \]

Write \( X = \bigsqcup_{i=1}^n V_i \), for \( V_i \) nonsingular, in any way.

**Definition**: Chern-Schwartz-MacPherson class

\[ c_{SM}(X) := \sum_i c_*(\mathbb{1}_{V_i}) \]

(Clearly Lego-like!)

**Theorem** (—, 2006)

*This is independent of all choices, and agrees with Schwartz/MacPherson’s definition.*
Two proofs:

- Using MacPherson’s result, easy exercise.

Classes de Chern pour variétés singulières, revisitée,

- Not using MacPherson’s natural transformation, prove directly
  that $c_*$ satisfies the Grothendieck-Deligne conjecture.

Limits of Chow groups and a new construction of
(MacPherson Volume 2), 915–941.

Useful side-product: functoriality with respect to not necessarily
proper morphisms, for an ‘enlarged’ Chow functor.
\( X \): a subscheme of a nonsingular variety \( M \).

**Definition:** Chern-Fulton class

\[
c_F(X) := c(TM) \cap s(X, M)
\]

**Example:** \( X \) a hypersurface in \( M \), then

\[
c_F(X) := c(TM) \cap (c(N_X M)^{-1} \cap [X]) = c(TM) \cap \frac{[X]}{1 + X}.
\]

Possibly better name for this: ‘virtual Chern class’ of \( X \).

If \( X \) is nonsingular, \( c_F(X) = c(TX) \cap [X] \).

Morally, \( c_F(X) \) is ‘the Chern class of a smoothening of \( X \)’.

A precise statement of this type: Fantechi-Göttsche 2007, Theorem 4.15.
Remark: the virtual class is not Lego-like. In particular, $c_F(X) \neq c_{SM}(X)$ in general. Link between $c_F(X)$, $c_{SM}(X)$: reasonably well-understood for hypersurfaces, complete intersections. (Work of many people.)

**Yokura:** The difference is captured by Verdier-Riemann-Roch-type results. (Close to Grothendieck’s motivation!)

**Definition:** Milnor class (up to sign...)

$c_F(X) - c_{SM}(X)$

- If $X$ is nonsingular, Milnor class $= 0$.
- If $X$ is a hypersurface, then $\pm \int c_F(X) - c_{SM}(X) = \text{sum of Parusiński-Milnor numbers of singularities.}$ (Hence the name.)
- In general, a quantification of ‘how singular $X$ is’.
Recall from 10 minutes ago:
The aim is to study ‘graph hypersurfaces’, in the Grothendieck group and from the point of view of singularities.

Γ: graph; one variable $t_e$ for each edge $e$
(Usually assume Γ is connected and 1–PI: it cannot be disconnected by removing a single edge.)

**Definition**

$$\Psi_\Gamma(t) = \sum_{T \subseteq \Gamma} \prod_{e \notin E(T)} t_e$$
where the sum is over all the spanning trees $T$ of $\Gamma$.

# of variables = # (internal) edges; degree = # loops
Example: $\Gamma =$ $n$-sided polygon

List all spanning trees, and edges *missed* by the spanning trees:

$$\psi_{\Gamma} = t_1 + t_2 + \cdots + t_n.$$
Example: ‘banana graphs’: two vertices, \( n \) parallel edges

\[
\psi_\Gamma = t_2 t_3 + t_1 t_3 + t_1 t_2 \text{ for } n = 3.
\]

\[
\psi_\Gamma = t_1 \cdots t_n \left( \frac{1}{t_1} + \cdots + \frac{1}{t_n} \right)
\]
\( \{ \psi_\Gamma = 0 \} \): hypersurface \( X_\Gamma \subseteq \mathbb{P}^{n-1} \) (or \( \hat{X}_\Gamma \subseteq \mathbb{A}^n \)); \( \deg X_\Gamma = \ell \).

\( n = \text{number of edges of } \Gamma; \ell = \text{number of loops.} \)

**Task:** compute the class \([X_\Gamma]\) in the Grothendieck ring, and/or \( c_{SM}(X_\Gamma) \in A_*\mathbb{P}^{n-1}. \)

Equivalent: \([\mathbb{P}^{n-1} \setminus X_\Gamma], c_{SM}(\mathbb{P}^{n-1} \setminus X_\Gamma) \in A_*\mathbb{P}^{n-1}. \)

(Closer to motivation: the Feynman amplitude of \( \Gamma \) is a period of the complement of \( X_\Gamma \).)

For example: \( \chi(\mathbb{P}^{n-1} \setminus X_\Gamma) =? \)

Relation between these invariants and combinatorics of \( \Gamma \)?
For instance, does $\chi(\mathbb{P}^{n-1} \setminus X_\Gamma)$ closely reflect the combinatorics of $\Gamma$?

Devil’s advocate (= referee to CNTP 2009): maybe not too closely. Indeed, $\chi(\mathbb{P}^N \setminus X_{\Gamma_1 \Pi \Gamma_2}) = 0$. (Reason: $\mathbb{C}^*$-action.)

Challenge: Beyond computing invariants for individual graphs, understand the organization of these invariants for all graphs. (This is in fact necessary in order to approach renormalization issues.)
In low dimension, $c_{\text{SM}}$ classes may be computed with e.g. Macaulay2.

http://www.math.fsu.edu/~aluffi/CSM/CSMexamples.html

Experimentation for small graphs: J. Stryker, almost all graphs with six or fewer edges.

Puzzle: $c_{\text{SM}}(X_\Gamma)$ is effective for all these graphs! Why?

Evokes:

- $c_{\text{SM}}(T)$ is effective for all toric varieties. (”Ehlers’ formula”)
- $c_{\text{SM}}(S)$ is conjecturally effective for all Schubert varieties $S$ of ordinary Grassmannians (— & Mihalcea, JAG 2009)

Note these are all mixed-Tate...
Infinite families of graphs?

**Theorem** (—, Marcolli, CNTP 2009)

*Explicit computation of* \([X_\Gamma] \in \text{Grothendieck ring, and } c_{\text{SM}}(X_\Gamma), for \Gamma = \text{all banana graphs}*

In the Grothendieck group:

\[
[\mathbb{P}^{n-1} \setminus X_{\Gamma_n}] = \frac{T^n - (-1)^n}{T + 1} + nT^{n-2}
\]

\[
= T^{n-1} + (n - 1)T^{n-2} + T^{n-3} - T^{n-4} + T^{n-5} + \cdots \pm 1
\]

where \(T = [A^1 \setminus A^0] = L - 1\).

\((L = [A^1])\)
The CSM class:

\[ c_{SM}(\mathbb{I}_{\mathbb{P}^{n-1} \setminus X_{\Gamma}}) = ((1 - H)^{n-1} + nH) \cap [\mathbb{P}^{n-1}] \]

where \( H \) is the hyperplane class in \( \mathbb{P}^{n-1} \).

‘Large’ Milnor class (↔ ‘very singular’). Example, \( n = 9 \):
\[
84H^3 - 1176H^4 + 9786H^5 - 78792H^6 + 630516H^7 - 5044200H^8
\]

Corollary: \( \chi(X_{\Gamma}) = n + (-1)^n \) for \( n \geq 3 \).

In particular, \( \chi(X_{\Gamma}) > 0 \) for all banana graphs. In fact, \( c_{SM}(X_{\Gamma}) \) is effective for banana graphs.
Proof of the theorem:

If $\Gamma$ is any planar graph, can relate $X_\Gamma$ to $X_{\Gamma^\vee}$, where $\Gamma^\vee$ is the dual graph: they correspond to each other via a Cremona transformation of $\mathbb{P}^{n-1}$.

For $\Gamma =$ banana graphs:
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For $\Gamma =$ banana graphs:
\[ \Gamma^\vee \] are polygons, computation can be carried out explicitly. (Calculus of constructible functions, and lemma on \( c_{SM} \) classes via ‘adapted blow-ups’.)

Remark: More generally, one expects certain sums of \([X_\Gamma]\) to be ‘easier’ (and more interesting) than individual \([X_\Gamma]\).

**Bloch, 2008:** computation of \( \sum [X_\Gamma] \), \( \Gamma \) connected graph with \( N \) vertices (with automorphism factor); it is \( \text{MT} \). Main tool: the relation between \([X_\Gamma]\) and \([X_{\Gamma^\vee}]\), extended to non-planar graphs.
Reason why $\Gamma$ assumed to be connected, 1–PI: Integrals $U(\Gamma, p)$ are multiplicative on disjoint unions of graphs. If $\Gamma = \Gamma_1 \amalg \Gamma_2$, then

$$U(\Gamma, p) = U(\Gamma_1, p_1)U(\Gamma_2, p_2)$$

If $\Gamma$ is obtained by joining $\Gamma_1$, $\Gamma_2$ by an edge (matching external momenta), multiply product by a ‘propagator’ term.

**FEYNMAN RULES!**

With Marcolli: ‘Algebro-geometric Feynman rules’
(I vetoed ‘Feynman rules in algebraic geometry’)

Back to the challenges presented earlier:

**Challenge:** Understand the organization of invariants such as $[\mathbb{P}^{n-1} \setminus X_\Gamma], \ c_{SM}(\mathbb{P}^{n-1} \setminus X_\Gamma)$ for all graphs. Understand relation between the combinatorics of a graph and the corresponding invariants.

Ways to formalize these:

- Give formulas for the behavior of invariants after combinatorial operations such as splitting edges, adding edges. . .
- Look for ‘Feynman rules’ based on the class in the Grothendieck ring and on $c_{SM}$ classes.

First task: some formulas are obtained in CNTP 2009. Second task: maybe more interesting.
The following recipe is part of a larger picture:

- $\Gamma$: finite graph (may be non-connected, non-1-PI...), $n$ edges
- $\hat{X}_\Gamma$: corr. hypersurface in $\mathbb{A}^n$; view as locally closed in $\mathbb{P}^n$
- $c_*(\mathbb{I}_{\hat{X}_\Gamma}) = a_0[\mathbb{P}^0] + \cdots + a_n[\mathbb{P}^n]$
- Define $G_\Gamma(T) = a_0 + a_1 T + \cdots + a_n T^n$
- Define $C_\Gamma(T) = (T + 1)^n - G_\Gamma(T)$

Example: $\Gamma = \text{banana graph} \leadsto C_\Gamma(T) = T(T - 1)^{n-1} + nT^{n-1}$

Remarks:
- Coefficient of $T^{n-1}$ in $C_\Gamma(T)$ equals $n - \ell$.
- $C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \setminus X_\Gamma)$. 
Theorem (—, Marcolli, arXiv:0811.2514)

The invariant $C_\Gamma(T)$ obeys the Feynman rules, with inverse propagator $(T + 1)$.

Proof:
Show that Feynman rules correspond to homomorphisms from a ‘Grothendieck ring’ of conical immersed subvarieties of $\mathbb{A}^n$. The function $G_\Gamma(T)$ is such a homomorphism. Proof of this fact: study $c_{SM}$ classes of joins in projective space.

⇝ ‘Feynman rules’ for $c_{SM}$ classes of graph hypersurfaces are a particular case of behavior of $c_{SM}$ classes with respect to natural constructions in projective geometry.
Note that this answers the objection on $\chi(\mathbb{P}^{n-1} \setminus X_\Gamma)$: This is one coefficient of $C_\Gamma(T)$; it is not multiplicative under disjoint union, but $C_\Gamma(T)$ is.

Similar story at the level of motives:

$\Gamma \leadsto [\mathbb{A}^n \setminus \hat{X}_\Gamma]$.

**Theorem** (—, Marcolli, arXiv:0811.2514)

*This invariant also satisfies the Feynman rules, with inverse propagator $L = [\mathbb{A}^1]$.*

In arXiv:0811.2514, we obtain a ‘universal’ invariant.
More recent work with Marcolli: a possible approach to explaining the BK evidence. (Reference: arXiv:0901.2107.)

Idea: Transfer the integral computation to a fixed variety $D_\ell$ (for given number $\ell$ of loops) $\rightsquigarrow$ for all graphs with $\ell$ loops, the Feynman integral is a period of a fixed $D_\ell$ relative to a locus $S_\ell$ supported on strata of a fixed normal crossing divisor.

Here, $D_\ell$ is the complement of the determinant hypersurface, clearly MT.

The translation holds for graphs satisfying reasonable combinatorial conditions, e.g.: 3-vertex connected, each vertex admits a wheel neighborhood.
This reduces the question to ‘linear algebra’: describe a variety of frames \((v_1, \ldots, v_\ell)\) with \(v_1 \in V_1, \ldots, v_\ell \in V_\ell\), where \(V_1, \ldots, V_\ell\) are (arbitrary) subspaces of a fixed vector space.

Prove this is MT!

Ravi Vakil: This is bound to be hard.
(‘Murphy’s law in algebraic geometry’)

Low \(\ell\) (=few loops): fun exercise.

Example: \(V_1, V_2\): arbitrary subspaces of a fixed vector space \(V\); 
\(\mathcal{F}(V_1, V_2) = \) variety of pairs \((v_1, v_2)\) s.t. \(v_i \in V_i\), and \((v_1, v_2)\) linearly independent.

\([\mathcal{F}(V_1, V_2)] = ??\)
\[ d_i = \dim V_i; \quad d_{12} = \dim (V_1 \cap V_2): \]

\[
\mathbb{F}(V_1, V_2) = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}
\]

\[
\ell = 3, \text{ notation as above (} \delta = \dim (V_1 + V_2 + V_3)): \]

\[
\mathbb{F}(V_1, V_2, V_3) = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)
- (\mathbb{L} - 1) \left( (\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1) \right)
+ (\mathbb{L} - 1)^2 \left( \mathbb{L}^{d_1+d_2+d_3-\delta} - \mathbb{L}^{d_{123}+1} \right) + (\mathbb{L} - 1)^3
\]

In particular, both are mixed-Tate. (Both from arXiv:0901.2107.)

\[ \ell = 4: \text{ some work by J. Fullwood}; \text{ but it gets very messy very fast.} \]

\[ \mathbb{F}(V_1, \ldots, V_r) \text{ may be expressed as an intersection of Schubert varieties in flag manifolds; these tend to be very complex gadgets.} \]

(And remember: intersections of MT are not necessarily MT!)
SUMMARY:

- Numerical evidence suggests that individual contributions of graphs to Feynman integrals may be ‘very special’ numbers.
- One way to approach this question is to study certain (very) singular varieties associated to graphs.
- Classes in the Grothendieck group and characteristic classes are natural ways to quantify ‘how singular’ these varieties are.
- It turns out that these invariants satisfy the ‘Feynman rules’, a natural set of constraints in the theory of Feynman integrals.
- A new approach reduces the question to the study of certain varieties of frames, with relations to e.g. the geometry of Schubert varieties in flag manifolds.

Just two more things...
Contents
Introduction
Crash course on Chern/Milnor classes of singular varieties
Invariants of graph hypersurfaces
Two more things

http://www.journalofsingularities.org/

Paolo Aluffi
Chern classes of singular varieties and Feynman integrals
HAPPY BIRTHDAY, ANATOLY!