Invariants of Normal Surface Singularities

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assume: M is a **rational homology sphere** ($H_1(M, \mathbb{Z})$ is finite)

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One has an embedding $L\hookrightarrow L'$ with $L'/L=H_1(M,\mathbb{Z})$

Notation: $H:=H_1(M,\mathbb{Z})$, $\widehat{H}$ = Pontjagin dual of H

$$\theta: H\to\widehat{H} \text{ natural isomorphism } [l']\mapsto\theta([l']):=e^{2\pi i(l',\cdot)}$$

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Problem: determine these invariants
Question: when are they topological ? (computable from Γ or M)

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Problem: determine these invariants
Question 1: what are their peculiar/additional properties (for a singularity link M)?
Question 2: how they influence the analytic invariants?

Cohomology of line bundles

Most of the analytic geometry of $\widetilde{X}$ (hence of (X, o) too) is described by its line bundles and their cohomology groups.

Basic problem: ***For any*** $\mathcal{L} \in \mathrm{Pic}(\widetilde{X})$ ***and effective cycle*** $l \in L_{\geq 0}$ ***recover the dimensions***

$$(a)\ \dim \frac{H^0\left(\widetilde{X}, \mathcal{L}\right)}{H^0\left(\widetilde{X}, \mathcal{L}(-l)\right)} \quad \textbf{\textit{and}} \quad (b)\ \dim H^1(\widetilde{X}, \mathcal{L})$$

from the combinatorics of Γ (at least for some families of singularities).

Natural line bundles

For arbitrary line bundles (and when $Pic^0(\widetilde{X})\neq 0$) this question can be very hard.

There is an increasing optimism to understand this problem for 'special' line bundles: the **natural** line bundles.

They are provided by the splitting of the cohomological exponential exact sequence:

we associate canonically to each $l'\in L'=H^2(\widetilde{X},\mathbb{Z})$
a line bundle on $\widetilde{X}$ whose first Chern class is l':

Natural line bundles

$$\begin{tikzcd}
 & & & L \arrow[dl, dashed] \arrow[d] & \\
0 \arrow[r] & H^1(\widetilde{X},\mathcal{O}_{\widetilde{X}}) \arrow[r] & \mathrm{Pic}(\widetilde{X}) \arrow[r, shift left, "c_1"] & L' \arrow[l, dashed, shift left, "\mathcal{O}"] \arrow[r] & 0.
\end{tikzcd}$$

The first Chern class c_1 is surjective and it has an obvious section on the subgroup L: it maps every element to its associated line bundle: $l \mapsto \mathcal{O}(l)$

This section has a *unique extension* $\mathcal{O}$ to L'. We call a line bundle *natural* if it is in the image of this section:

$$l' \mapsto \mathcal{O}(l')$$

Natural line bundles

Example of natural line bundle:
Let $c:(Y,o)\to(X,o)$ be the **universal abelian cover** of (X,o), $\pi_Y:\widetilde{Y}\to Y$ the normalized pullback of π by c, $\widetilde{c}:\widetilde{Y}\to\widetilde{X}$ the morphism which covers c. The action of H on (Y,o) lifts to $\widetilde{Y}$ and one has an H-eigenspace decomposition

$$\widetilde{c}_*\mathcal{O}_{\widetilde{Y}}=\bigoplus_{l'\in Q}\mathcal{O}(-l'),$$

where $\mathcal{O}(-l')$ is the $\theta([l'])$-eigenspace of $\widetilde{c}_*\mathcal{O}_{\widetilde{Y}}$. This is compatible with the eigenspace decomposition of $\mathcal{O}_{Y,o}$ too.

Above: $Q=\{\sum l'_vE_v\in L',\ 0\leq l'_v<1\}$.

Equivariant Hilbert series

Once a resolution π is fixed, $\mathcal{O}_{Y,o}$ inherits the *divisorial multi-filtration*

$$\mathcal{F}(l') := \{f \in \mathcal{O}_{Y,o} \,|\, \mathrm{div}(f \circ \pi_Y) \geq \widetilde{c}^*(l')\}.$$

$\mathfrak{h}(l')$ = dimension of the $\theta([l'])$-eigenspace of $\mathcal{O}_{Y,o}/\mathcal{F}(l')$.
The *equivariant divisorial Hilbert series* is

$$\mathcal{H}(\mathbf{t}) = \sum_{l'=\sum l_v E_v \in L'} \mathfrak{h}(l') t_1^{l_1} \cdots t_s^{l_s} = \sum_{l' \in L'} \mathfrak{h}(l') \mathbf{t}^{l'} \in \mathbb{Z}[[L']].$$

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$\sum_{l \in L} \mathfrak{h}(l)\mathbf{t}^l =$ H-invariants of $\mathcal{H}$
Hilbert series of the $\pi^{-1}(o)$-divisorial multi-filtration of $\mathcal{O}_{X,o}$

Second central problem: ***Recover*** $\mathcal{H}(\mathbf{t})$ ***from*** Γ (for some families of singularities).

The Campillo–Delgado–Guzein-Zade series

$$\mathcal{P}(\mathbf{t}) = -\mathcal{H}(\mathbf{t}) \cdot \prod_v (1 - t_v^{-1}) \in \mathbb{Z}[[L']].$$

(Above, $\mathbb{Z}[[L']]$ is regarded as a module over $\mathbb{Z}[L']$.)

Note: $\mathcal{P}(\mathbf{t})$ nd $\mathcal{H}(\mathbf{t})$ determine each other.

Campillo, Delgado and Gusein-Zade for rational singularities proposed a *topological description for* $\mathcal{P}(\mathbf{t})$.

Question: ***how general is this topological characterization, and where are its limits***?

Some results:

Theorem [N.] With the notation $E_I = \sum_{v \in I} E_v$, consider

$$h_{\mathcal{L}} := \sum_{I \subseteq \mathcal{V}} (-1)^{|I|+1} \dim \frac{H^0(\mathcal{L})}{H^0(\mathcal{L}(-E_I))},$$

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Then

$$\dim \frac{H^0(\mathcal{L})}{H^0(\mathcal{L}(-l))} = \sum_{a\in L_{\geq 0}, a\not\geq l} h_{\mathcal{L}(-a)}.$$

This, for any $l'\in L'$ implies

$$\mathfrak{h}(l') = \sum_{a\in L,\, a\not\geq 0} h_{\mathcal{O}(-l'-a)}.$$

$$\mathcal{P}(\mathbf{t}) = \sum_{l'\in L'} h_{\mathcal{O}(-l')}\mathbf{t}^{l'}.$$

Some results:

Moreover,
there exists a constant $\mathrm{const}_{[\mathcal{L}]}$, depending only on the class of $[\mathcal{L}] \in Pic(\widetilde{X})/L$, such that

$$-h^1(\mathcal{L}) = \sum_{a \in L,\, a \not\leq 0} h_{\mathcal{L}(a)} + \mathrm{const}_{[\mathcal{L}]} + \frac{(K - 2c_1(\mathcal{L}))^2 + |\mathcal{V}|}{8}.$$

(Above: $K =$ canonical class in L', $|\mathcal{V}| =$ number of vertices in Γ.)

Corollary:

The invariants

$$\dim \frac{H^0(\mathcal{L})}{H^0(\mathcal{L}(-l))} \quad \text{and} \quad h^1(\mathcal{L})$$

(for natural line bundles), and the Hilbert series $\mathcal{H}(\mathbf{t})$ are determined by

$$\mathcal{P}(\mathbf{t}) \qquad \text{and} \qquad \{\mathrm{const}_h\}_{h\in H}.$$

Question: ***When are*** $\mathcal{P}(\mathbf{t})$ ***and*** $\{\mathrm{const}_h\}_{h\in H}$ ***topological ?***

Positive result:

Theorem [N.] If (X, o) is splice quotient singularity (including rational, minimally elliptic, weighted homogeneous singularities), then

$$\mathcal{P}(\mathbf{t}) = \prod_{v\in\mathcal{V}} \left(1 - \mathbf{t}^{E_v^*}\right)^{\delta_v - 2},$$

$(E_v^* \in L'$ dual basis: $(E_v^*, E_w) = -\delta_{v,w}$; δ_v= degree of vertex v).

$$\mathrm{const}_h = \mathsf{sw}_{h*\sigma_{\mathrm{can}}},$$

the Seiberg–Witten invariant of M associated with the *spin*^{c}–structure $\sigma = h * \sigma_{can}$.
($Spin(M)$ is an H–torsor, with a canonical element σ_{can}.)

Principal cycles:

Principal cycles $\mathcal{P}r$ = it consists of the restrictions to E of the divisors of π-pullbacks of analytic functions from $\mathcal{O}_{X,o}$.

Principal $\mathbb{Q}$–cycles: $\mathcal{P}r'$ = those rational cycle $l' \in L'$ for which $\mathcal{O}(-l')$ has a global holomorphic section which is not zero on any of the exceptional components. $(\mathcal{P}r' \cap L = \mathcal{P}r)$

Natural topological background for $\mathcal{P}r$ and $\mathcal{P}r'$:

$$\mathcal{S}' := \{l' \in L' \,:\, (l', E_v) \leq 0 \text{ for all } v \in \mathcal{V}\}.$$

and

$$\mathcal{S} := \mathcal{S}' \cap L \qquad \text{(Lipman's cone)}$$

Fact: $\mathcal{P}r$ (resp. $\mathcal{P}r'$) sub-semigroup of $\mathcal{S}$ (resp. $\mathcal{P}r$)

Problem: ***Find $\mathcal{P}r$ and $\mathcal{P}r'$.***

Are they topological (for some singularities)?

Principal cycles:

Theorem [N.] Assume that (X, o) is splice-quotient. Then $\mathcal{P}r'$ (hence $\mathcal{P}r = \mathcal{P}r' \cap L$ too) is topological.

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Then $l' \in \mathcal{P}r'$ iff there exists finitely many monomial cycles $\{D(\alpha_{(k)})\}_k \in l' + L$ so that $l' = \inf_k D(\alpha_{(k)})$.

$\pi^* m_{X,o}$, multiplicity

Theorem [N.] Assume that (X, o) is splice-quotient, $m_{X,o}$ the maximal ideal of $\mathcal{O}_{X,o}$, and write

$$\pi^* m_{X,o} = \mathcal{O}_{\tilde{X}}(-Z_{max}) \otimes \bigotimes_{P \in \mathcal{B}} \mathcal{I}_P$$

(Z_{max}= maximal (ideal) cycle, $\mathcal{B}$ base points.)

Then, Z_{max} and $\{\mathcal{I}_P\}_{p\in\mathcal{B}}$ can be characterized topologically. Moreover, there is an explicit closed combinatorial formula for the multiplicity too.

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(Rational singularities: Artin; minimally elliptic: Laufer; elliptic: N.)

Heegaard Floer homology

Expectation: Besides the 'Seiberg–Witten invariant formula', there is a deeper, and more complex relation at the level of Heegaard–Floer homology (of Ozsváth–Szabó)

$HF^+(M,\sigma)$ is a $\mathbb{Z}[U]$-module, with two gradings (a $\mathbb{Z}_2$ and a $\mathbb{Q}$–grading), depending on $\sigma \in Spin^c(M)$.

In the sequel we assume σ is the 'canonical *spin*^{c}–structure (induced by the complex analytic structure, but it can be identified topologically too).

Lattice (co)homology

The bridge between singularity invariants and Heegaard–Floer theory is realized by the **lattice cohomology** (introduced by the author).

$M =$ singularity link, $\sigma \in Spin^c(M)$, $q \geq 0$ integer:

We define: $\mathbb{H}^q(M, \sigma)$, a $\mathbb{Z}[U]$–module.

(we will consider here only $\sigma = \sigma_{can}$)

Lattice cohomology

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Each $l \in L$ and $I \subset \mathcal{V}$ with $|I| = k$ determines a $\square_k$ = a k–dimensional cube in $L \otimes \mathbb{R}$, which has its vertices in the lattice points $(l + \sum_{j \in I'} E_j)_{I'}$, where I' runs over all subsets of I.

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For any $n \in \mathbb{Z}$: S_n= union of all the cubes (of any dimension) $\square_k$, such that

$$\chi(\text{any vertex of } \square_k) \leq n$$

Lattice cohomology

Definition:

$$\mathbb{H}^q(\Gamma) := \oplus_n H^q(S_n, \mathbb{Z})$$

$\mathbb{Z}[U]$–module structure (U–action): restriction

$$\ldots \to H^q(S_{n+1}, \mathbb{Z}) \to H^q(S_n, \mathbb{Z}) \to \ldots$$

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Examples: $\mathbb{H}^*$ is computed for any ‘almost rational graph’ (including rational, elliptic and ‘star–shaped graphs’). Rational and elliptic singularities can be characterized via $\mathbb{H}^*$.

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Conjecture: $\mathbb{H}^*(M)$ contains a lot of information about the analytic structure.