Homomorphisms between Kähler groups (Jaca)

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Introduction

Compact Kähler manifolds (and in particular smooth projective varieties) are special!

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A Kähler group is the fundamental group of a compact Kähler manifold, such as a smooth projective variety.

Kähler groups

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- (A.-Nori, Delzant) Most solvable groups are not Kähler
- (Carlson-Toledo, Simpson) Many lattices in semisimple Lie groups are not Kähler
- (Gromov-Schoen, A.-Bressler-Ramachandran) "Big" groups, such as free products, are not Kähler

Conjecture

A basic open question in the theory

Does the class of Kähler groups = the class of fundamental groups of smooth projective varieties?

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(Unfortunately, Voisin's examples don't give anything here.)

Kähler homomorphisms

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Lemma

If $h : \Gamma_1 \to \Gamma_2$ is Kähler then the image, kernel and cokernel of the induced map

$$\Gamma_1/[\Gamma_1, \Gamma_1] \to \Gamma_2/[\Gamma_2, \Gamma_2]$$

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has even rank.

Proof.

The Hodge structure on homology is functorial. □

Kähler homomorphisms (examples)

Recall

Theorem (Serre)

Any finite group is Kähler

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Lemma (Botong Wang)

Any homomorphism from a Kähler group to a finite group is Kähler.

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If Γ is Kähler , then the canonical map

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Proof.

Use the Albanese. □

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Let

$$\Gamma_g = \langle a_1, \ldots a_{2g} \mid [a_1, a_{g+1}] \ldots [a_g, a_{2g}] = 1 \rangle$$

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By work of Beaville-Siu

Proposition

A surjective homomorphism $h : \Gamma \to \Gamma_g$ from a Kähler group is Kähler-surjective if doesn't factor through a larger surface group.

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Lemma

If $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ is Kähler, then its eigenvalues lie in an imaginary quadratic field.

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Lemma

If $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ is Kähler, then its eigenvalues lie in an imaginary quadratic field.

Proof.

f can be realized as an endomorphism of an elliptic curve. The argument is now standard. □

Splitting Obstruction

Say that a surjective homomorphism of groups $h : H \to G$ splits if it has a right inverse.

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The class $e(h) \in H^2(G, K/[K,K])$ associated to the extension

$$0 \to K/[K,K] \to H/[K,K] \to G \to 1$$

where $K = \ker(h)$, gives an obstruction to splitting.

Splitting theorem

A Kähler-surjective homomorphism between Kähler groups is a homomorphism induced by a surjective holomorphic map with connected fibres between compact Kähler manifolds. (Note Kähler-surjective $\Rightarrow$ surjective.)

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The main result for this talk is

Theorem

If h is Kähler-surjective then $e(h) \otimes \mathbb{Q} = 0$.

Applications

This leads to new obstructions for a group to be Kähler . For example, the group

$$
\begin{aligned}
\Gamma &= \langle a_1, \ldots a_{2g}, c \mid [a_1, a_{g+1}] \ldots [a_g, a_{2g}] = c, [a_i, c] = 1 \rangle \\
&= \text{extension } 1 \to \mathbb{Z} \to \Gamma \to \Gamma_g \to 1 \\
&\quad \text{classfied by the generator of } H^2(\Gamma_g, \mathbb{Z})
\end{aligned}
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is not Kähler.

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is not Kähler. (When $g = 1$, this is a Heisenberg group, and this follows from earlier obstructions.)

Proof.

If Γ is Kähler , then the map $h : \Gamma \to \Gamma_g$ is also Kähler . But $e(h) \otimes \mathbb{Q} \neq 0$. □

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- When $f : X \to Y$ admits a section, then $e(\pi_1(f)) = 0$.
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- When $f : X \to Y$ admits a section, then $e(\pi_1(f)) = 0$.
- When $f : X \to Y$ is a map of smooth projective varieties, then it admits a multisection, because the generic fibre has a rational point over a finite extension of $\mathbb{C}(Y)$.
- Therefore $e(\pi_1(f)) \otimes \mathbb{Q} = 0$.

Proof (general)

This doesn't work when $f : X \to Y$ is analytic, so we use a different strategy.

Proof (general)

Step 1: Since $\ker(\pi_1(f))$ may be nonfinitely generated, replace Y by a suitable orbifold where f has better structure.

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- Then identify $e(\pi_1(f))$ as lying in the image of a differential of a (perverse) Leray spectral sequence. for f.

Decomposition theorem

Step 3: Use M. Saito's decomposition theorem to show that this differential, and therefore $e(\pi_1(f))$, is zero.

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  A (perverse) sheaf on an orbifold is locally an equivariant sheaf on the manifold.

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- Theorem (Saito+ϵ)

  Suppose that $f : X \to Y$ is a proper holomorphic map of orbifolds with X Kähler. Let L be a perverse sheaf of geometric origin on X. Then

$$\mathbb{R}f_*L = \bigoplus_j IC(M_j)[m_j]$$

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- Corollary

  Perverse Leray degenerates ($\Rightarrow$ splitting theorem).

Thanks, and Happy Birthday Anatoly